Chapter II

Transmission Lines - Wave Equations

Electromagnetic Fields and Waves

In this Chapter:

Transmission Lines Overview

Examples of Transmission Lines: controlled geometry lines
Circuit component size compared to operating wavelength, λ
TL Analysis is needed if size is not too small compared to wavelength

Circuit Theory Analysis of Controlled Geometry Lines

RLCG Physically based Model (linear) – Distributed parameters
Use KVL and KCL to develop voltage-current relationships
Analysis in both Time and Frequency Domains
Second order differential equation for voltage as function of position and time

Solution in Frequency Domain

$V(z) = V^+e^{-\gamma z} + V^-e^{+\gamma z}$

$I(z) = (1/Z_o) \cdot \{V^+e^{-\gamma z} - V^-e^{+\gamma z}\}$

$Z_o = \sqrt{\{R_z + j\omega L_z\}/\{G_z + j\omega C_z\}}$

$\gamma = \alpha + j\beta = \sqrt{\{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\}}$

Harmonic solution in Time Domain

Using Laplace Transform

$v(z,t) = |V^+|e^{-\alpha z} cos(\omega t - \beta z + \varphi_v^+)$
$+|V^-|e^{+\alpha z} cos(\omega t + \beta z + \varphi_v^-)$

$i(z,t) = |V^+/Z_o|e^{-\alpha z} cos(\omega t - \beta z + \varphi_i^+)$
$-|V^-/Z_o|e^{+\alpha z} cos(\omega t + \beta z + \varphi_i^-)$

Transient solution in Time Domain

Only for Lossless Line $R_z=G_z=0$

$v(z,t) = v^+(t - \sqrt{L_zC_z}z) + v^-(t + \sqrt{L_zC_z}z))$

$i(z,t) = 1/\sqrt{L_z/C_z}\{v^+(t - \sqrt{L_zC_z}z) - v^-(t + \sqrt{L_zC_z}z)\}$

Physical Insight

Two traveling waves: "+z" wave and "–z" wave both have same Z_o and γ
Z_o: Characteristic impedance, Ohms -- γ =α+jβ: Complex propagation constant, m^{-1}
α: Attenuation coefficient, Nepers/m -- β: Phase constant, rad/m
Phase velocity c_{ph}=ω/β, m/s -- Wavelength λ=c_{ph}/f=2π/β, m
$V^+$and V^- are determined by boundary conditions at TL source and load ends

Reflection Coefficient

Define Reflection coefficient Γ(z)= $(V^-e^{+\gamma z})/(V^+e^{-\gamma z})$ ="-z wave" /"+z wave"

$\Gamma(z) = \Gamma(\ell) \cdot e^{+2\gamma(z-\ell)}$ -- $\Gamma(\ell) = \Gamma_L = [Z_L - Z_o]/[Z_L + Z_o]$

$\Gamma(z) = [Z(z) - Z_o]/[Z(z) + Z_o]$ -- $Z(z) = Z_o[1 + \Gamma(z)]/[1 - \Gamma(z)]$

Reflections are caused by impedance discontinuities "mismatch – not equal to Zo"

Traveling and Standing Waves

$Z_L=Z_o$, $\Gamma_L=0\rightarrow\Gamma(z)=0\rightarrow Z(z)= Z_o$: $V^-=0$, only "+z wave", pure traveling wave, $|V(z)|= |V^+|e^{-\alpha z}$

Magnitude constant except for decay due to attenuation (loss)

$Z_L=0+jX_L$, $|\Gamma_L|=1\rightarrow|\Gamma(z)|=1\rightarrow Z(z)= 0+jX(z)$: $|V^+|= |V^-|$, pure standing waves, $|V(z)|=2|V^+|cos(\beta z+ \varphi)$

Wave does not travel, Magnitude has peaks (=2|V+|) and valleys of zeroes (nodes)

$Z_L\neq Z_o$, $\Gamma_L\neq0\rightarrow\Gamma(z)\neq0\rightarrow Z(z)\neq Z_o$: combination of "+z" and reflected "-z" waves
Mix of traveling {$|V^+|-|V^-|$}, and standing {$2|V^-|$} waves
Standing Wave Ratio, SWR = V_{max}/V_{min}=[1+|Γ|]/[1+|Γ|] -- |Γ|=0→SWR=1, |Γ|=1→SWR=∞

Reflections: Power and Signal Delivery

Reflections (standing waves) reduce power delivered to load – also degrade signal quality

Mismatch causes dispersion to communication signals –Reflections limit bandwidth of digital communications

For optimum power & signal delivery, impedance matching at both source and load ends is necessary

Addenda

A. Driving Point Impedance

$$Z(z) = Z_o \frac{[1+\Gamma(z)]}{[1-\Gamma(z)]} = Z_o \frac{\{Z_L + Z_o \tanh[\gamma(\ell - z)]\}}{\{Z_o + Z_L \tanh[\gamma(\ell - z)]\}}$$

For Lossless Lines:

$$\tanh[\gamma(\ell - z)] \rightarrow j\tan[\beta(\ell - z)] \text{ – ALSO: } Z(z \pm n_{int}\lambda/2) = Z(z) \;\&\; Z(z \pm n_{odd}\lambda/4) = R_o^2/Z(z)$$

Z_L	Γ_L	Z_{in}	Z_L	Γ_L	Z_{in}	Z_L	Γ_L	Z_{in}
Z_o (Matched)	0	Z_o	0 (Short Circuit)	-1	$j Z_o \tan(\beta\ell)$	∞ (Open Circuit)	+1	$-j Z_o \cot(\beta\ell)$

B. Impedance Matching

- *Matching Networks: L-section, T and Π using lumped elements work for low RF frequencies*
- *TL matching network using series/shunt stubs needed at higher RF frequencies*
  - *Add TL section at load "far" end – length chosen to yield reactive input impedance at "near" end*
  - *Add TL stub (series or shunt) to tune out the obtained reactance at the "near" end*
  - *Preferred stub: short-circuited at its far end and connected in shunt at its near end*

C. Frequency Domain Bounce Diagram

- *Unconventional approach to build steady state frequency domain solution through monitoring transients*
- *Initial launch by source: voltage divider between the "transient" input impedance of the line (Z_o) and the source impedance*
- *Initial launch is monitored as it travels down the line ($e^{-\gamma z}$), reflects off the load, Γ_L and reflects back off the source side Γ_s. Multiple reflections cumulate to form a geometric series expression.*

D. Time Domain Bounce Diagram

- *Conventional approach to monitor multiple reflection transients in the time domain. Networks with lossless lines and resistive networks are presented in this chapter*
- *Initial launch by source: voltage divider between the "transient" input resistance of the line (R_o) and the source resistance*
- *Initial launch is monitored as it travels down the line (time delay=z/c_{ph}), reflects off the load, ρ_L (local version of Γ_L) and reflects back off the source side ρ_s (local version of Γ_s). Multiple reflection cumulate to form a sequence of successful delays.*
- *Time Domain Reflectometry, TDR, is a practical tool for locating and characterizing cable faults and discontinuities.*

E. The Smith Chart

- *A graphical tool to solve TL impedance problems.*
- *Two-way transform between the normalized complex impedance, Zn,/admittance, Yn, (real and imaginary) and the complex reflection coefficient (real and imaginary or magnitude and phase).*
- *$Z_n = Z/Z_o = r+jx$, $Y_n = Y/Y_o = g+jb$, $\Gamma = \Gamma_{re} + j\,\Gamma_{im} = |\Gamma| \underline{/\varphi_\Gamma}$*
- *The chart maps Γ_{im} vs Γ_{re} with r and x (g and b) as parameters:*
  - *constant r (or g) circles have their origins on Γ_{im}=0 axis and all pass by the Γ=1+j0*
  - *constant x (or b) circles have their origins on Γ_{re}=1axis and all pass by the Γ=1+j0*
  - *the phasor stretching from the origin to the (r,x) intercept represent the corresponding Γ phasor. The phasor to the (g,b) intercept is the -Γ phasor*
  - *Scales are provided to read Γ magnitude (also re and im) as well as $\underline{|/\varphi_\Gamma}$*
  - *scale is provided to read the SWR*
- *The trace of Γ(z) is a circle centered at Γ=0+j0 for lossless lines and a spiral for lossy lines. Spiral pitch depends on the attenuation coefficient α*
- *Scales for normalized distance Δz/λ are given to guide moving from one line location to another on this trace for both directions of motion: Towards the Load and Towards the Generator (source)*
- *A full rotation around the chart correspond to a distance of half wavelength*
- *Applications of the Smith Chart are given:*
  - *find Γ given Z or Y and vice versa*
  - *find $Z(z_2)$, $Y(z_2)$, and $\Gamma(z_2)$ given $\{Z(z_1), Y(z_1), \text{ or } \Gamma(z_1)\}$ and $\{z_2-z_1\}$*
  - *stub impedance matching problems*

Transmission Lines:

Transmission Lines, TL, is the name we use to refer to those wires and cables that connect electrical and electronic devices (parts or components) to each other. The wires used to connect the light bulb to the electrical mains is a transmission line, the wire that connects the landline phone to the wall outlet is a transmission line, and the cable that connects the TV to the satellite receiver, the wall outlet, or antenna is also a transmission line. The wires you used in the circuits and electronics laboratories, and the wires you used in "bread-boarding" an analog or a digital circuit are all transmission lines. The conducting strips on (and within) the motherboard of the computer as well as the "interconnects" inside the semiconductor chips (processor, memory, etc.) are also transmission lines. Figure 2.1 demonstrates a few samples of different kinds of Transmission Lines.

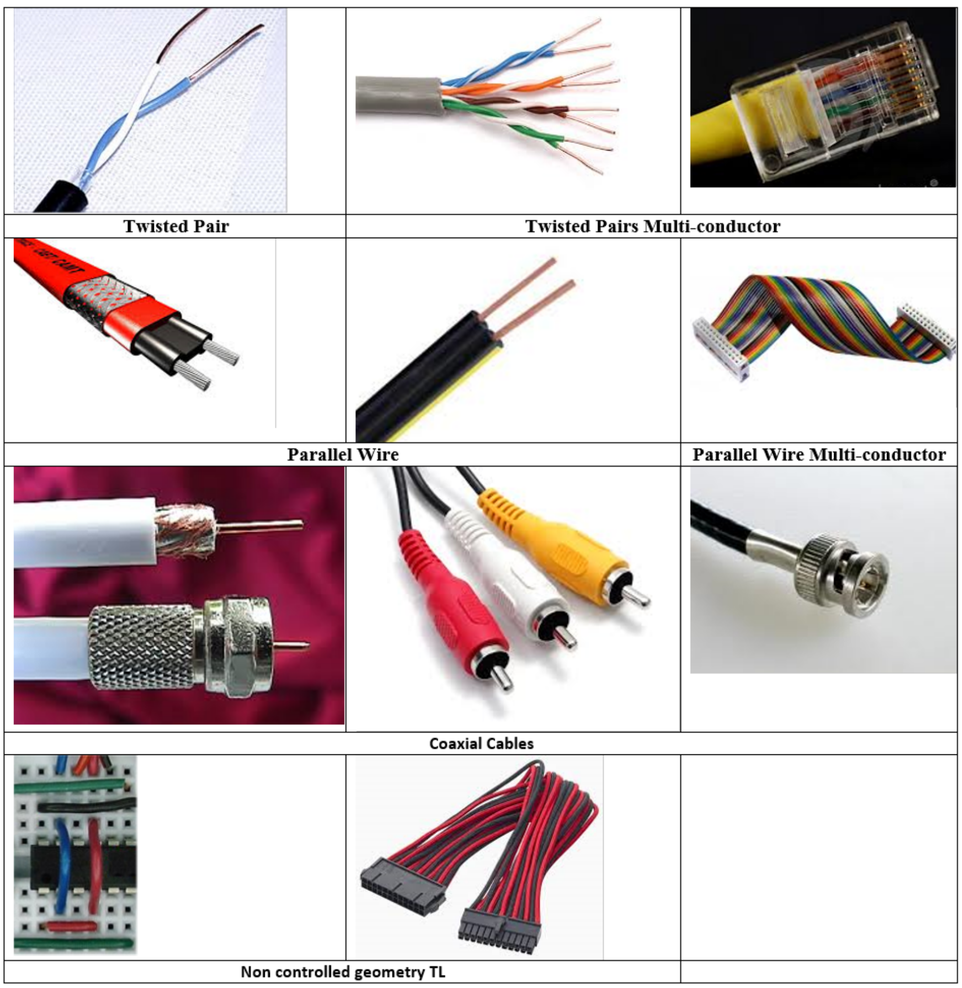


Figure 2.1

"Ok. So, what is so important about transmission lines to make this topic the opening chapter of an EM book?" We used those wires in our circuits and electronics labs with no issues whatsoever. We also wired so many of these "bread-boards" and things worked well. The answer is that it has to do with the length of the wire relative to the wavelength of the signals propagating, as well as the wire's internal resistance. Let us take these two items one at a time. First, we talk about wire length and wavelength.

To remind you about wavelength, we visualize a signal/wave propagating in one dimension, see Figure 2.2. Well, as the figure shows, the signal travels a distance as time elapses. The distance between two consecutive locations of similar phase on the wave (e.g. two consecutive peaks) is called the wavelength, λ. Simple physics tells us that that distance $\lambda = c/f$, where "c" is the wave's phase velocity and "f" is its frequency.

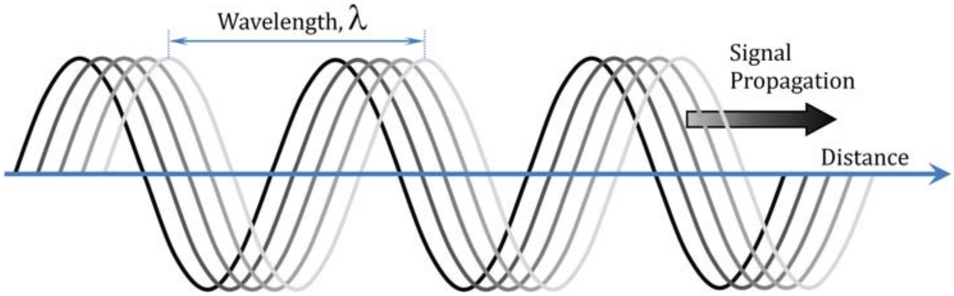


Figure 2.2

Now, let us assume that this signal propagates along a conducting wire (cable/transmission line) of length ℓ. Figures 2.3 demonstrates that the signal phase varies along the TL length. The amount of phase variation along the TL length, ℓ, depends on the length of this wire relative to the wavelength, λ. The ratio ℓ/ λ is the same as (*Δ phase /360°*).

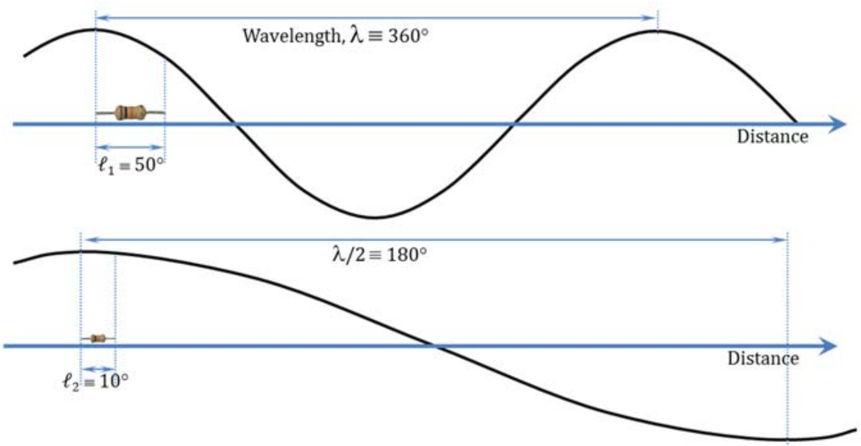


Figure 2.3

Hence, the change in signal phase between that at the beginning of the TL segment ℓ and its end, Δ phase, equals 360 * ℓ/λ. This change in phase corresponds to a change in signal amplitude (see the figure), and hence the signal at the beginning of the TL segment ℓ and its end are not identical. Consequently, if you are using this line to connect the output side of a device or a component and the input of another, then, one cannot assume that the signal that came out the first device entered into the second. Rather one should account for the changes introduced by the interconnecting TL.

Working at low frequencies, the wavelength is long and most typical wiring arrangements are too short compared to it. In these cases, the Δ phase is infinitesimally small and so is the change in signal amplitude. [Notice that for DC "signals" (zero frequency), the wavelength is infinite]

Now, we turn briefly to the second item, "the wire's internal resistance" which may not be ignored even if ℓ/λ is infinitesimally small. At low frequencies and at DC, we would only be concerned about the signal drop as a result of the wire's internal resistance.

Back to the cases where the ratio ℓ/λ matters. The question is how much ℓ/λ can we tolerate (ignore) in our analysis (design)? The answer to this question lies in deciding how much signal level (and hence phase) change can be tolerated. Considering the wave changes the fastest around its zero crossing where $sin(x) \approx x$, then the worst-case analysis holds. Table 2.1 demonstrates the orders of magnitudes of circuit dimensions that can be tolerated at different operating frequencies. Three "tolerance" levels are given corresponding to different levels of phase errors.

Table 2.1

% Change in Signal Amplitude [x]	*10.0%*	*1.0%*	*0.1%*	
Corresponding Δ Phase in Radians [$sin^{-1}(x)$]	*0.100*	*0.010*	*0.001*	
Corresponding Δ Phase in Degrees	*5.739*	*0.573*	*0.057*	
Corresponding ℓ/λ [Δ Phase/2π]	*0.016*	*0.0016*	*0.00016*	

Frequency	*Wavelength in air*	*Tolerated Length (air)*	*Tolerated Length (air)*	*Tolerated Length (air)*	*Metric Unit*
1 Hz	*300.0*	*4.78264*	*0.47747*	*0.04775*	*Mm*
60 Hz	*5.0*	*0.07971*	*0.00796*	*0.00080*	*Mm*
1 kHz	*300.0*	*4.78264*	*0.47747*	*0.04775*	*km*
1 MHz	*300.0*	*4.78264*	*0.47747*	*0.04775*	*m*
10 MHz	*30.0*	*0.47826*	*0.04775*	*0.00477*	*m*
100 MHz	*3.0*	*0.04783*	*0.00477*	*0.00048*	*m*
1 GHz	*30.0*	*4.78264*	*0.47747*	*0.04775*	*mm*
10 GHz	*3.0*	*0.47826*	*0.04775*	*0.00477*	*mm*
100 GHz	*3.0*	*0.04783*	*0.00477*	*0.000477*	*mm*
1 THz	*0.3*	*4.78264*	*0.47747*	*0.04775*	*μm*

Frequency	*wavelength*	*Tolerated Length (air)*	*Tolerated Length (air)*	*Tolerated Length (air)*	*English Unit*
1 Hz	*186,411*	*2,971.79600*	*296.68784*	*29.66829*	*mile*
60 Hz	*3,107*	*49.52993*	*4.94480*	*0.49447*	*mile*
1 kHz	*186*	*2.97180*	*0.29669*	*0.02967*	*mile*
1 MHz	*0.186*	*0.00297*	*0.00030*	*0.00003*	*mile*
10 MHz	*98.4*	*1.56911*	*0.15665*	*0.01566*	*feet*
100 MHz	*9.84*	*0.15691*	*0.01567*	*0.00157*	*foot*
1 GHz	*11.8*	*0.18829*	*0.01880*	*0.00188*	*inch*
10 GHz	*1.18*	*0.01883*	*0.00188*	*0.000188*	*inch*
100 GHz	*0.118*	*1.8829*	*0. 18829*	*0.018829*	*mil*
1 THz	*0.0118*	*0. 18829*	*0.018829*	*0.0018829*	*mil*

So, as you can see, if you are building a circuit (or device) to operate at 1 GHz, your wiring interconnects should be shorter than 0.04775 cm (0.0188"), otherwise you must analyze the performance of such interconnect using "Transmission Line Theory" to avoid errors more than 1%.

Transmission Line Analysis (Theory):

The accurate and comprehensive way to analyze the performance of transmission lines is to do the analysis as an electromagnetic fields problem. This requires the use of Maxwell's equations to develop a full EM field model and study the field waves as they propagate along the TL. This process will be the subject of Chapter XIII of this book. This means that there is a lot for us to learn before we can get into such analysis. However, we can use a circuit theory approach to develop a circuit model and get a simplified analysis for the TL for the time being.

The interesting point here is that the circuit theory tools that we will be using to develop TL analysis are not but the derivatives of the field theory and Maxwell's equations. We learned these tools in earlier circuit courses and we are familiar with them; specifically, Ohm's Law and Kirchhoff's Voltage and Current Laws.

Representing a TL by an equivalent circuit containing circuit elements that physically models the TL requires future knowledge that will be developed in later chapters as well. However, let us assume that we guess what these circuit elements are like (qualitatively) and have some physical insight of their nature. For example, we can guess that the line conductors do have some internal resistance (R), the TL as a "loop" of two conductors has magnetic "inductive" behavior (L), the two conductors separated by a dielectric represent a capacitance (C), and the imperfect dielectric isolation between the two conductors would cause leakage "conduction" (G). While we are familiar with these components in circuit analysis, the EM field definition and physical insight of these components will be the subject of later chapters in this book (Capacitance, Chapter VI – Inductance, Chapter IX, and resistance and Conductance, Chapter VI).

Circuit Theory Analysis of a Two Conductor Controlled Geometry TL:

In this book, we will confine our study of TL theory to the simplest of cases; that of two conductors with uniform cross section (material and geometry do not vary along the TL length). This is what we call controlled geometry two conductor transmission line. The lines will be assumed straight with no curves or bends.

We will further limit ourselves to the case where the TL material, i.e. both conductors and the insulating dielectric(s), are linear (properties do not change with field intensity), and frequency independent (properties do not change with frequency).

Examples of such TLs include the parallel wire TL, the coaxial TL, and the microstrip TL, see Figure 2.4.

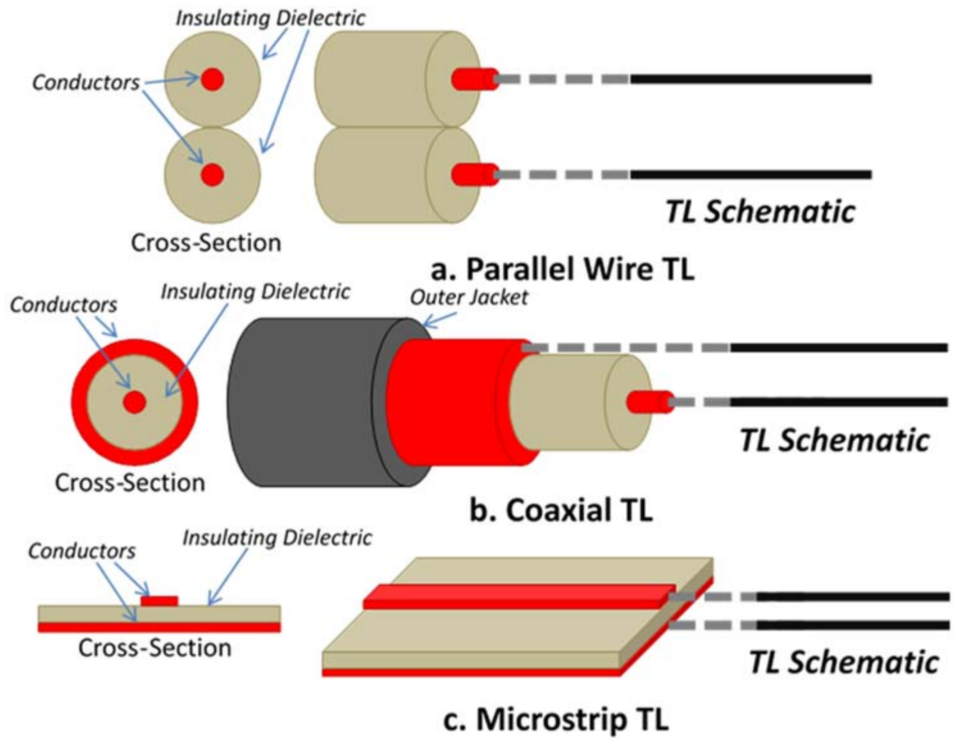


Figure 2.4: Examples of Controlled Geometry TL

RLCG Model:

To develop a physically based model, we have to realize that it is a distributed one, i.e. it is distributed over the length of the line and is not lumped at any specific location. To explain, we talk about the internal resistance of the two conductors forming the line. That resistance is determined by the conductor material, the cross-sectional area of the conductor, and the length of the wire. The resistance is distributed along the length of the wire and is not lumped at either end or anywhere else. So, we can model it as a distributed resistance and we can define a measure "R_z" as the resistance per unit length of the transmission line. This would be the sum of the total "circuit" resistance per unit length of both TL conductors. The units of R_z would be Ohms/m.

Likewise, we can talk about the distributed inductance of the TL circuit. We will use L_z to denote the inductance per unit length, Henry/meter. For the insulating dielectric(s), we model two different physical parameters; C_z (Farads/meter) and G_z (Siemens/meter). The capacitance per unit TL length, C_z, is the physical model of the dielectric(s) insulation between the two conductors and the conductance per unit TL length, G_z, is the physical model of the imperfect insulation of the dielectric causing leakage current through the dielectric. Figure 2.5 depicts the nature of these RLCG elements. This physically based circuit model of the TL is not perfect or accurate, but it will help us set our feet into this mysterious domain of TL theory. This circuit model is typically referred to as the "RLCG" model.

It is to be noted that most references name the distributed RLCG parameters as R, L, C , and G with no subscript. This practice may lead to some confusion as it implies a reference to "whole" resistance, inductance, capacitance and conductance quantities and not the per unit length parameters. We choose to add the "z" subscript to distinguish these quantities as per unit "z" length parameters. Hence, the incremental "whole" resistance of a line increment of length Δz would be $\Delta R=[R_z][\Delta z]$. Likewize, $\Delta L=[L_z][\Delta z]$, $\Delta C=[C_z][\Delta z]$, and $\Delta G=[G_z][\Delta z]$ for the inremental "whole" inductance, capacitance, and conductance, respectively.

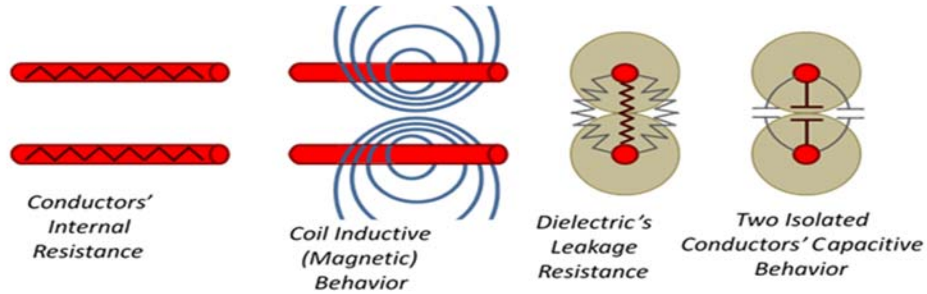


Figure 2.5

Transmission Line Circuit Analysis Using the Distributed RLCG Model:

For this analysis, we will assume the line to extend along a linear axis, let us call it the z-axis, Figure 2.6. At z=0 we would have the input side (or the sending/transmitter end) to the TL (typically, a signal source with an internal "impedance") and at the end of the line, say $z=\ell$, the line output side, a load would be connected (receiving/receiver end).

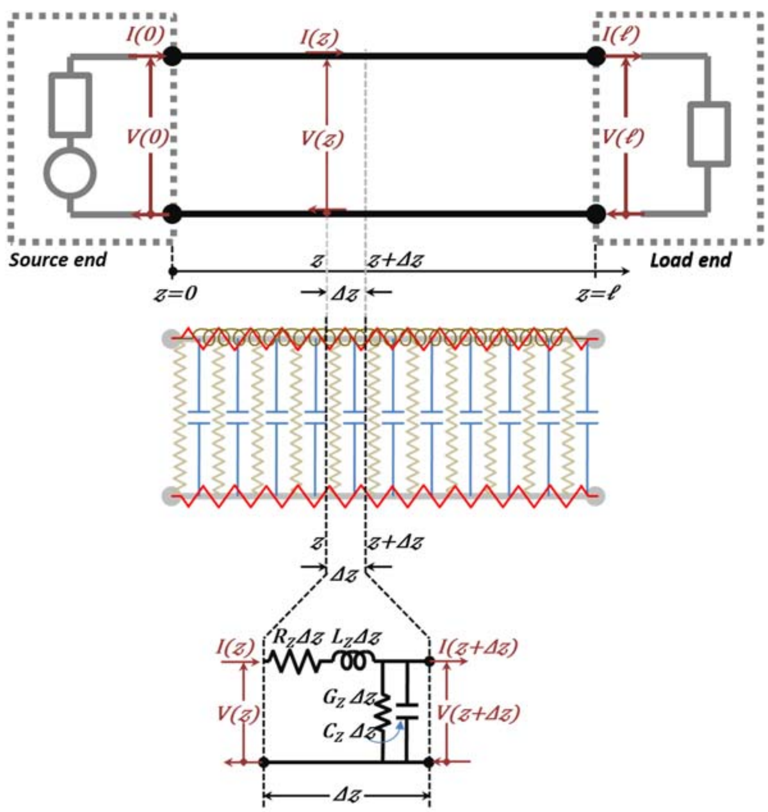


Figure 2.6

Now, as we replace the line by its distributed RLCG circuit equivalency, we expect the voltage and currents to vary with position along the line. We will use the notation *V(z)* and *I(z)*, for the voltage and current phasors as functions position, respectively, [or $v(z,t)$ and $i(z,t)$ for instantaneous (time domain) expressions]. We use the upper case notations for the phasors in the frequency domain analysis, while the lower case represent the time domain variables.]. At z=0 and at $z=\ell$, the voltage and current terms would be *V(0)*, *I(0)*, *V(ℓ)*, and *I(ℓ)*, [or $v(0,t), i(0,t), v(\ell,t)$, $and\ i(\ell, \mathrm{t})$], respectively.

Next, we need to develop expressions for the voltages current relationships as functions of position and time. In order to develop a thorough and insightful analysis of the TL network, we intend to carry the analysis in both the time and frequency domains simultaneously. Moreover, we will also be monitoring the solutions for both the transient and the steady (harmonic analysis) states as we go. As we take the analysis back and forth between the two domains, some readers may find it a bit confusing. However, going over this section a couple of times, and with focused attention, the results will be all positive.

Steady-State Harmonic Analysis:

In the following, we will start with the steady-state harmonic analysis in both time and frequency domains. We will assume an excitation in the form of a time harmonic voltage

$$v_s(t) = |V_s|\, cos(\omega t + \varphi_s) \underset{freq\ dom}{\Longleftrightarrow} V_s(j\omega) = |V_s| e^{j\varphi_s}$$

Since we are limiting our discussion to a linear system (without any nonlinear elements), then all solution voltages and currents are expected to be of the same frequency as that of the source; for example, the voltage on the line at a general location z at time t would assume the following form:

$$v(z,t) = |V(z)|\, cos(\omega t + \varphi(z)) \underset{freq\ dom}{\Longleftrightarrow} V(z,j\omega) = |V(z)| e^{j\varphi(z)}$$

> ***In the following, we will develop TL equations and solutions in both frequency and time domains side by side. The left column would be for the time domain development while the right one will be for the frequency domain. In doing so, we are attempting not to get "disoriented/lost in the ocean of the frequency domain and lose our ability to get back to shore". Instead, we will constantly relate to the physics of the development and the solution by "swimming close to shore" and monitoring the time domain phenomena while doing the frequency domain development.***

To analyze the "electric circuit" shown, we need to build the distributed equation for *V(z, jω)* and *I(z, jω)* [or $v(z,t)\ and\ i(z,t)$] along the line. We do that by considering an infinitesimally small increment of the line length *Δz* (to start at *z* and end at *z+Δz*). Writing the circuit (input-output) relationships between the voltages and currents for the *Δz* increment enables us to build the differential equation describing the behavior of this physically based circuit model. To do that, we use Kirchhoff's Voltage and Kirchhoff's Current Laws (KVL and KCL) for the *Δz* "equivalent" circuit (see Figure 2.6).

Time Domain Analysis: $\boldsymbol{v(z,t)}$ and $\boldsymbol{i(z,t)}$	*Eq #*	*Frequency Domain Analysis:* $\boldsymbol{V(z,j\omega)}$ and $\boldsymbol{I(z,j\omega)}$
Forming the Differential Equations:		
We start with the differential relationship, while skipping the writing of ",t" and "jω" in all voltage and current expressions		
$v(z+\Delta z,t) = v(z,t) + \Delta v(z,t)$	*(2.1)*	$V(z+\Delta z) = V(z) + \Delta V(z)$
$i(z+\Delta z,t) = i(z,t) + \Delta i(z,t)$	*(2.2)*	$I(z+\Delta z) = I(z) + \Delta I(z)$
Now, we apply KVL		
$v(z,t) = v_R + v_L + [v(z+\Delta z,t)]$	*(2.3)*	$V(z) = V_R + V_L + [V(z+\Delta z)]$
Where v_R and v_L [V_R and V_L] are the voltage drops across $R_z\,\Delta z$ and $L_z\,\Delta z$, and using line (2.1)		
$v(z,t) = (R_z\,\Delta z)\cdot i(z,t) + (L_z\,\Delta z)\cdot\frac{di(z,t)}{dt} + [v(z,t) + \Delta v(z,t)]$	*(2.4)*	$V(z) = (R_z\,\Delta z)\cdot I(z) + (j\omega L_z\,\Delta z)\cdot I(z) + [V(z) + \Delta V(z)]$
$\frac{\Delta v(z,t)}{\Delta z} = -\left\{R_z\cdot i(z,t) + L_z\cdot\frac{di(z,t)}{dt}\right\}$	*(2.5)*	$\frac{\Delta V(z)}{\Delta z} = -\{R_z\cdot I(z) + j\omega L_z\cdot I(z)\}$
Taking the limit as Δz becomes infinitesimally small and approaches zero		
$\frac{dv(z,t)}{dz} = -\left\{R_z\cdot i(z,t) + L_z\cdot\frac{di(z,t)}{dt}\right\}$	*(2.6)*	$\frac{dV(z)}{dz} = -\{R_z + j\omega L_z\}\cdot I(z)$
Next, we apply KCL		
$i(z,t) = i_G + i_C + [i(z+\Delta z,t)]$	*(2.7)*	$I(z) = I_G + I_C + [I(z+\Delta z)]$
Where i_G and i_C [I_G and I_C] are the current drops through $G_z\,\Delta z$ and $C_z\,\Delta z$, and using Equation (2.2)		
$i(z,t) = (G_z\Delta z)\cdot v(z+\Delta z,t) + (C_z\Delta z)\cdot\frac{dv(z+\Delta z,t)}{dt} + [i(z,t) + \Delta i(z,t)]$	*(2.8)*	$I(z) = (G_z\Delta z)\cdot V(z+\Delta z) + (j\omega C_z\Delta z)\cdot V(z) + [I(z) + \Delta I(z)]$
$\frac{\Delta i(z,t)}{\Delta z} = -\left\{G_z\cdot v(z+\Delta z,t) + C_z\cdot\frac{dv(z+\Delta z,t)}{dt}\right\}$	*(2.9)*	$\frac{\Delta I(z)}{\Delta z} = -\{G_z\cdot V(z+\Delta z) + j\omega C_z\cdot V(z+\Delta z)\}$
Taking the limit as Δz becomes infinitesimally small and approaches zero		
$\frac{di(z,t)}{dz} = -\left\{G_z\cdot v(z,t) + C_z\cdot\frac{dv(z,t)}{dt}\right\}$	*(2.10)*	$\frac{dI(z)}{dz} = -\{G_z + j\omega C_z\}\cdot V(z)$
Solution:		
We have two simultaneous differential equations in i and v. So, we need to combine them in a way to end up with a new equation in one variable only.		
Differentiate line (2.6) with respect to (z) and line (2.10) with respect to (t)		*Differentiate line (2.6) with respect to z and using line (2.10)*
$\frac{d^2v(z,t)}{dz^2} = -\left\{R_z\cdot\frac{di(z,t)}{dz} + L_z\cdot\frac{d^2i(z,t)}{dtdz}\right\}$	*(2.11)*	$\frac{d^2V(z)}{dz^2} = -\{R_z + j\omega L_z\}\cdot\frac{dI(z)}{dz}$
$\frac{d^2i(z,t)}{dzdt} = -\left\{G_z\cdot\frac{dv(z,t)}{dt} + C_z\cdot\frac{d^2v(z,t)}{dt^2}\right\}$	*(2.12)*	
Combining lines (2.11) and (2.12)		*Combining lines (2.10) and (2.11)*
$\frac{d^2v(z,t)}{dz^2} = -\left\{R_z\cdot -\left\{G_z\cdot v(z,t) + C_z\cdot\frac{dv(z,t)}{dt}\right\} + L_z\cdot -\left\{G_z\cdot\frac{dv(z,t)}{dt} + C_z\cdot\frac{d^2v(z,t)}{dt^2}\right\}\right\}$	*(2.13)*	

$$\frac{d^2v(z,t)}{dz^2} = \left\{R_zG_z \cdot v(z,t) + [R_zC_z + L_zG_z] \cdot \frac{dv(z,t)}{dt} + L_zC_z \cdot \frac{d^2v(z,t)}{dt^2}\right\}$$	*(2.14)*	$$\frac{d^2V(z)}{dz^2} = \{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\} \cdot V(z)$$
Hence, here we have one equation in one variable; the voltage. It is important to recognize that the resulting Equation (2.14) is a second order differential equation, D.E. We combined two first order D.E. to yield one second order one. *In doing so, we had to differentiate one of the two first order D.E. Taking the derivative eliminates the constant term in that quantity. This elimination means that we lost one piece of information and that information is not represented as part of the obtained second order D.E. The only way to recover that missing information is by using the original first order D.E. before differentiation.* *Therefore, what we need to do is solve (2.14) for the voltage and get back to Equation (2.6) to get the current.*		
To solve the D.E. in (2.14), we use the Laplace Transform		
$$\frac{d^2V(z)}{dz^2} = \{R_zG_z \cdot V(z) + [R_zC_z + L_zG_z] \cdot sV(z) + L_zC_z \cdot s^2V(z)\}$$		
$$\frac{d^2V(z,s)}{dz^2} = \{R_z + sL_z\} \cdot \{G_z + sC_z\} \cdot V(z)$$	*(2.15)*	$$\frac{d^2V(z,j\omega)}{dz^2} = \{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\} \cdot V(z)$$
Defining γ^2		
$\gamma^2(s) = R_zG_z + s[R_zC_z + L_zG_z] + s^2L_zC_z$	*(2.16)*	$\gamma^2(j\omega) = \{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\}$
$$\frac{d^2V(z)}{dz^2} = \gamma^2 \cdot V(z)$$	*(2.17)*	$$\frac{d^2V(z)}{dz^2} = \gamma^2 \cdot V(z)$$
Solving the DE in line (2.17)		
$V(z) = V^+e^{-\gamma z} + V^-e^{+\gamma z}$	*(2.18)*	$V(z) = V^+e^{-\gamma z} + V^-e^{+\gamma z}$
Where V^+ and V^- are constants		
Laplace Transform applied to line 6		*From line 6*
$$\frac{dV(z)}{dz} = -\{R_z + sL_z\} \cdot I(z)$$	*(2.19)*	
$$I(z) = -\frac{1}{\{R_z + sL_z\}} \cdot \frac{dV(z)}{dz}$$	*(2.20)*	$$I(z) = -\frac{1}{\{R_z + j\omega L_z\}} \cdot \frac{dV(z)}{dz}$$
$$I(z) = \frac{\gamma}{\{R_z + sL_z\}} \cdot \{V^+e^{-\gamma z} - V^-e^{+\gamma z}\}$$	*(2.21)*	$$I(z) = \frac{\gamma}{\{R_z + j\omega L_z\}} \cdot \{V^+e^{-\gamma z} - V^-e^{+\gamma z}\}$$
$$\frac{\gamma}{\{R_z + sL_z\}} = \frac{1}{Z_o}$$	*(2.22)*	$$\frac{\gamma}{\{R_z + j\omega L_z\}} = \frac{1}{Z_o}$$
Discussion of Possible Solutions:		*Summary:*
The inverse Laplace transform of these equations back to the time domain in the general case is too complicated. However, possible solutions can be achieved for either the case of harmonic signals (s=jω) and the case of lossless TL ($R_z = 0$ and $G_z = 0$)	*(2.23)*	$$I(z) = \frac{1}{Z_o} \cdot \{V^+e^{-\gamma z} - V^-e^{+\gamma z}\}$$
	(2.24)	$Z_o = R_o + jX_o = \sqrt{\{R_z + j\omega L_z\}/\{G_z + j\omega C_z\}}$
	(2.25)	$\gamma = \alpha + j\beta = \sqrt{\{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\}}$
	(2.26)	$V(z) = V^+e^{-\gamma z} + V^-e^{+\gamma z}$
Case of steady state harmonic signals (s=jω): See Right Column starting line (2.23)	*(2.27)*	$V(z) = V^+e^{-\alpha z}e^{-j\beta z} + V^-e^{+\alpha z}e^{+j\beta z}$
	(2.28)	*Transfer Back to Time Domain:*
	(2.29)	$$v(z,t) = \lvert V^+\rvert e^{-\alpha z}cos(\omega t - \beta z + \varphi_v^+) + \lvert V^-\rvert e^{+\alpha z}cos(\omega t + \beta z + \varphi_v^-)$$
	(2.30)	$$i(z,t) = \lvert V^+/Z_o\rvert e^{-\alpha z}\,cos(\omega t - \beta z + \varphi_v^+ - \varphi_z) - \lvert V^-/Z_o\rvert e^{+\alpha z}cos(\omega t + \beta z + \varphi_v^- - \varphi_z)$$

Case of lossless TL ($\boldsymbol{R_z = 0}$ ***and*** $\boldsymbol{G_z = 0}$)

Lines (2.16) and (2.22) reduce to		*Lines (2.25) and (2.22) reduce to*
$\gamma^2(s) = s^2 L_z C_z$ and $Z_o = \sqrt{L_z/C_z}$	*(2.31)*	$\gamma = j\beta = j\omega\sqrt{L_z C_z}$ and $Z_o = \sqrt{L_z/C_z}$
Which reduces (2.18) and (2.21) to		*Which reduces (2.27) and (2.23) to*
$V(z) = V^+ e^{-s\sqrt{L_z C_z} z} + V^- e^{+s\sqrt{L_z C_z} z}$	*(2.32)*	$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$
$I(z) = \frac{1}{\left\{\sqrt{L_z/C_z}\right\}} \cdot \left\{V^+ e^{-s\sqrt{L_z C_z} z} - V^- e^{+s\sqrt{L_z C_z} z}\right\}$	*(2.33)*	$I(z) = \frac{1}{\sqrt{L_z/C_z}}\left\{V^+ e^{-j\beta z} - V^- e^{+j\beta z}\right\}$
Transfer Back to Time Domain:		*Also, (2.29) and (2.30) reduce to*
$v(z,t) = v^+(t - \sqrt{L_z C_z} z) + v^-(t + \sqrt{L_z C_z} z)$	*(2.34)*	$v(z,t) = \lvert V^+\rvert \cos(\omega t - \beta z + \varphi_v^+) + \lvert V^-\rvert \cos(\omega t + \beta z + \varphi_v^-)$
$i(z,t) = \frac{1}{\sqrt{L_z/C_z}}\left\{v^+\left(t - \sqrt{L_z C_z} z\right) - v^-\left(t + \sqrt{L_z C_z} z\right)\right\}$	*(2.35)*	$i(z,t) = \lvert V^+/Z_o\rvert \cos(\omega t - \beta z + \varphi_v^+) - \lvert V^-/Z_o\rvert \cos(\omega t + \beta z + \varphi_v^-)$

So, we have arrived at a solution for the steady-state harmonic case using both the time domain and the frequency domain approaches. The obtained solutions are expressed by Equations (2.29)-(2.30) in the time domain and in Equations (2.23)-(2.27) in the frequency-domain phasor format. We did not carry out the solution in the time domain; instead, we made use of the Laplace transform to solve the second order D.E. In a way, that was similar to resorting to the frequency domain for help. The Laplace domain is but a generalization of the frequency domain as it uses $s = \sigma + j\omega$ and hence allows the study of transients. When $\sigma = 0$, the Laplace domain coincides with the frequency domain and the problem reduces to its steady-state harmonic solution.

In the special case of a lossless TL, we were able to carry out the solution further in the time domain to non-harmonic cases. These solutions are given by Equations (2.34)-(2.35). The corresponding harmonic solutions are given on the right hand column of the same lines.

Physical Implications of Solutions - $\boldsymbol{\alpha, \beta, \lambda,}$ and $\boldsymbol{c_{ph}}$:

At this point, it is important to take a pause to examine the obtained results and make sure that we relate to their physical implications.

We examine the harmonic solution obtained in the time domain:

$$v(z,t) = \lvert V^+\rvert e^{-\alpha z} \cos(\omega t - \beta z + \varphi_v^+) + \lvert V^-\rvert e^{+\alpha z} \cos(\omega t + \beta z + \varphi_v^-)$$

This solution has two terms. Let us take them one at a time: $\lvert V^+\rvert e^{-\alpha z} \cos(\omega t - \beta z + \varphi_v^+)$ describes a voltage wave that changes with both t and z, a graph of which is shown in Figure 2.7

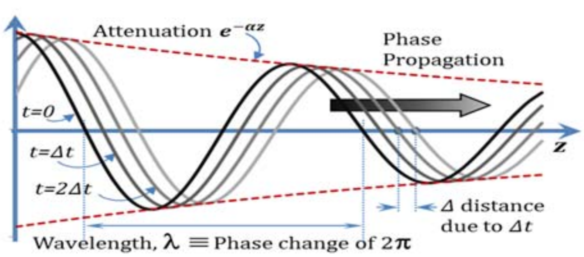


Figure 2.7

It is evident from the figure that this term represent a travelling wave that moves in the positive z direction (and hence the plus superscript in denoting V^+). The wave amplitude attenuates (decays) exponentially with distance with an attenuation coefficient of $e^{-\alpha z}$.

The wave experiences a full cycle in a distance λ, and hence the corresponding phase change $\beta\lambda$ should equal to 2π. i. e. $\beta\lambda = 2\pi$, or $\lambda = 2\pi/\beta$.

The velocity of its phase propagation can be derived by monitoring the motion of a specific phase on the wave,

$$\omega t_1 - \beta z_1 + \varphi_v^+ = \omega t_2 - \beta z_2 + \varphi_v^+ \textit{ and hence } \omega\Delta t = \beta\Delta z \textit{ , or } \Delta z/\Delta t = c_{ph} = \omega/\beta$$

$$\textit{Phase velocity}\, c_{ph} = \frac{\omega}{\beta} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f \qquad (2.36)$$

With this definition for the phase velocity, the terms of Equations (2.29) can be rewritten as:

$$v(z,t) = |V^+|\, e^{-\alpha z} \cos\left[\omega\left(t - \frac{z}{c_{ph}}\right) + \varphi_v^+\right] + |V^-|\, e^{+\alpha z} \cos\left[\omega\left(t + \frac{z}{c_{ph}}\right) + \varphi_v^-\right] \qquad (2.37)$$

which makes Equation (2.34) for the lossless line reduce to:

$$v(z,t) = |V^+| \cos\left[\omega\left(t - \frac{z}{c_{ph}}\right) + \varphi_v^+\right] + |V^-|\ \cos\left[\omega\left(t + \frac{z}{c_{ph}}\right) + \varphi_v^-\right] \qquad (2.38)$$

Now, comparing this form to that obtained in the time domain for the lossless line in Equation (2.34), we conclude that the time argument $\left(t - \sqrt{L_z C_z} z\right)$ must be equivalent to $\left(t - \frac{z}{c_{ph}}\right)$ obtained in Equation (2.38), which implies that

$$c_{ph} = \frac{1}{\sqrt{L_z C_z}} \qquad (2.39)$$

Consequently, the time domain solution in the lossless line case, Equation (2.34), becomes:

$$v(z,t) = v^+\left(t - \frac{z}{c_{ph}}\right) + v^-\left(t + \frac{z}{c_{ph}}\right) \qquad (2.40)$$

Now, back to the second term of the $v(z,t)$ solution $\left\{|V^-|e^{+\alpha z}\cos(\omega t + \beta z + \varphi_v^-) = |V^-|\cos\left[\omega\left(t + \frac{z}{c_{ph}}\right) + \varphi_v^-\right]\right\}$, Figure 2.8. This term represent a travelling wave that moves in the negative $(-z)$ direction (and hence the minus superscript in denoting V^-). The wave amplitude attenuates (decays) exponentially with distance with an attenuation coefficient of $e^{(-\alpha)(-z)} = e^{+\alpha z}$. This term has the same exact α *and* β of the V^+ term.

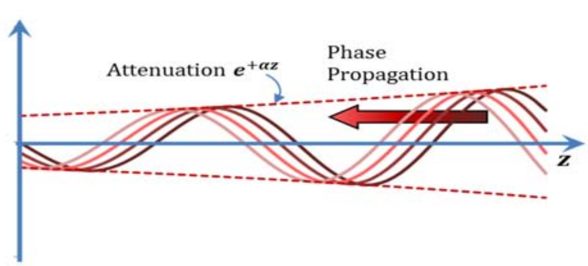


Figure 2.8

The two terms combined represent the $v(z,t)$ solution obtained in Equation (2.29). The complete solution is shown in Figure 2.9.

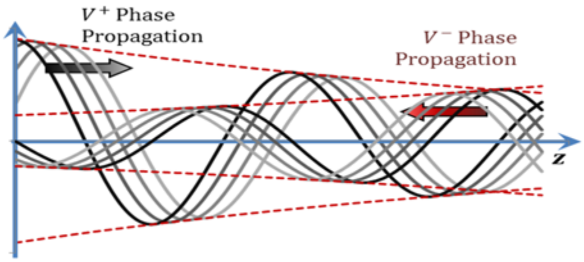


Figure 2.9

By the way, and before we leave this point, in the lossless TL case, the waves on the TL would not suffer any attenuation. The corresponding wave shapes are shown in Figure 2.10.

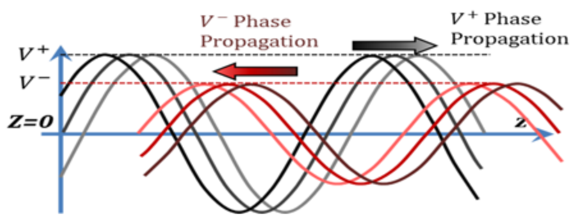


Figure 2.10

Physical Implications of Solutions - $\gamma, Z_o, V^+, and\ V^-$:

Now as we saw the physical meaning of $\alpha\ and\ \beta$, let us move back to the frequency domain solution in order to interpret the physical meaning of the parameters: $\gamma, Z_o, V^+, and\ V^-$.

$$\gamma = \alpha + j\beta = \sqrt{\{R_z + j\omega L_z\} \cdot \{G_z + j\omega C_z\}}$$

γ is a complex quantity with real part α and an imaginary part β .

The real part, α, appear in the exponent of the voltage and current solutions as to affect the amplitude of the wave along the TL (coordinate z). This real part becomes zero for lossless lines $R_z = G_z = 0$ and, hence, it is related with losses and causes the signals to attenuate as they travel along the line. For this reason, α will be called the *attenuation coefficient*, and its units are Nepers/meter. $\{One\ Np\ (Neper) = 20/ln(10) \cong 8.686\ dB\ (decibels)\}$

The imaginary part, β, also appears in the exponent of the voltage and current solutions as to affect the phase of the wave along the TL (coordinate z). Hence, β will be called the *phase constant*, and its units are radians/meter. For lossless lines, $\beta = \omega\sqrt{L_z C_z}$.

γ is called the *complex propagation constant* of the TL.

$$Z_o = R_o + jX_o = \sqrt{\{R_z + j\omega L_z\}/\{G_z + j\omega C_z\}}$$

Z_o is a complex quantity with real part R_o and an imaginary part X_o . For lossless lines, it reduces to a real value:

$$Z_o = R_o = \sqrt{L_z/C_z}$$

This may appear rather puzzling as we talk about a "Lossless" TL with R_o equal to so many Ohms, e.g., a cable TV TL has a characteristic impedance of 75 Ohms. The thing is that there are no 75 Ohms resistor(s) in existence anywhere on this line and the 75 Ohms does not represent its internal resistance. It simply means that the 75 Ohms is nothing but a characteristic number representing the ratio between the traveling voltage and current waves.

Z_o is called the *characteristic impedance* of the TL.

It is worthy to note that both Z_o *and* γ fully characterize the TL model. Given Z_o *and* γ and using Equations (2.24) and (2.25), the TL RLCG model can be fully determined.

While both Z_o *and* γ are characteristic parameters of the TL, V^+, *and* V^- are not. $V^+ = |V^+|e^{j\varphi_v^+}$, and $V^- = |V^-|e^{j\varphi_v^-}$ are two arbitrary constants that resulted from solving the TL differential equation without applying the boundary conditions, B.C. These constants can be fully determined once the TL conditions at its boundaries (z=0 and at $z=\ell$) are implemented.

Figure 2.11

Referring to Figure 2.11, the boundary conditions, B.C., at the Source end are given by:

$$V(0) = V^+ + V^- = V_s - Z_s \cdot I(0) = V_s - Z_s \cdot \frac{1}{Z_o}(V^+ - V^-),$$

and the B.C. at the Load end are

$$V(\ell) = V^+e^{-\gamma\ell} + V^-e^{+\gamma\ell} = Z_L \cdot I(\ell) = Z_L \cdot \left\{\frac{1}{Z_o}\left(V^+e^{-\gamma\ell} - V^-e^{+\gamma\ell}\right)\right\}$$

These are two "complex variable" equations that can be solved to yield both V^+ and V^- (magnitudes and phases). The resulting expressions can be written in the form:

$$V^+ = \left\{\frac{V_s\,Z_o}{Z_s+Z_o}\right\}\left\{\frac{1}{1-\left(\frac{Z_s-Z_o}{Z_s+Z_o}\right)\left(\frac{Z_L-Z_o}{Z_L+Z_o}\right)e^{-2\gamma\ell}}\right\} \qquad (2.41)$$

$$V^- = V^+\left(\frac{Z_L-Z_o}{Z_L+Z_o}\right)e^{-2\gamma\ell} \qquad (2.42)$$

Physically, these are the levels of the waves travelling on the line. As will be discussed later, the V^+ is the level of the voltage wave traveling in the positive z direction, while the V^- is that of the negative z wave. Considering that we only have one source of power (at the source end), there is only one explanation for the presence of a negative z traveling wave; that it must have resulted as a result of waves reflecting (bouncing back) from the load end. A useful analogy here is watching the water waves at the seashore. The sea is the source of water that travels towards the shore. After the water waves hit the shore, we notice water waves traveling from the shore back to the sea. These are reflections of the original sea waves.

Physical Implications of Solutions - Γ:

The V^- wave results from reflections of the V^+ wave as it bounces off the TL "discontinuity" at the load end. Hence, we define the ratio of the "reflected" (-z) wave to the "incident" (+z) wave as the reflection coefficient.

$V(z) = V^+e^{-\gamma z} + V^-e^{+\gamma z}$	*(2.18)*
$I(z) = \frac{1}{Z_o} \cdot \{V^+e^{-\gamma z} - V^-e^{+\gamma z}\}$	*(2.23)*
Defining $\Gamma(z)$	
$\Gamma(z) = \frac{V^-e^{+\gamma z}}{V^+e^{-\gamma z}} = \frac{V^-}{V^+}e^{+2\gamma z}$	*(2.43)*
$V(z) = V^+e^{-\gamma z}[1 + \Gamma(z)]$	*(2.44)*
$I(z) = \frac{V^+e^{-\gamma z}}{Z_o}[1 - \Gamma(z)]$	*(2.45)*
Hence, the "driving point impedance" at any location on the line, $Z(z)$ *is given by*	
$Z(z) = \frac{V(z)}{I(z)} = Z_o\frac{[1+\Gamma(z)]}{[1-\Gamma(z)]}$ *and hence,* $\Gamma(z) = \frac{[Z(z)-Z_o]}{[Z(z)+Z_o]}$	*(2.46)*
At both TL ends, $Z(z)$ *reduces to*	
$Z(0) = \frac{V(0)}{I(0)} = Z_o\frac{[1+\Gamma(0)]}{[1-\Gamma(0)]}$	*(2.47)*
$Z(\ell) = \frac{V(\ell)}{I(\ell)} = \{Z_L\} = Z_o\frac{[1+\Gamma(\ell)]}{[1-\Gamma(\ell)]}$	*(2.48)*
This gives us an insight regarding $\Gamma(\ell)$	
$Z_L = Z_o\frac{[1+\Gamma(\ell)]}{[1-\Gamma(\ell)]} \Rightarrow \Gamma(\ell) = \frac{[Z_L - Z_o]}{[Z_L + Z_o]}$	*(2.49)*
It means that $\Gamma(\ell)$ *exists as a result of the impedance discontinuity (mismatch)* $Z_L \neq Z_o$	
Using (2.43) twice at both z *and* ℓ	
$\Gamma(z) = \frac{V^-e^{+\gamma z}}{V^+e^{-\gamma z}}$ and $\Gamma(\ell) = \frac{V^-e^{+\gamma\ell}}{V^+e^{-\gamma\ell}}$	*(2.50)*
$\Gamma(z) = \Gamma(\ell) \cdot e^{+2\gamma(z-\ell)} = \Gamma(\ell) \cdot e^{-2\gamma(d)}$	*(2.51)*
Here, $d = \ell - z$ *represents a distance "d" starting at the load end and extending opposite to the "z" coordinate, see Figure 2.12.*	
The use of the coordinate "d" instead of "z" is typical in solving impedance matching problems in TL analysis and design	

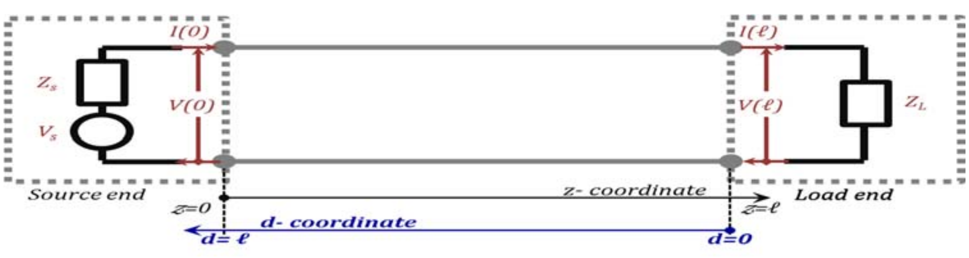


Figure 2.12

From Equation (2.49), we conclude that if it was not for the mismatch $Z_L - Z_o$ between the load impedance Z_L *and* the TL characteristic impedance Z_o, and as a consequence the reflection coefficient $\Gamma(z)$, the reflection (-z wave) $V^-e^{+\gamma z}$ would not exist. Therefore, we can say that waves traveling on a uniform TL with a matched end are fully absorbed by the load and will never experience any reflections. Likewise, if the TL is uniform and infinite in length, with no discontinuities, the source end launched waves would continue to travel indefinitely resulting in the absence of reflections (-z waves). This is exactly what happens to a light beam projected in the open space. If it meets a reflective object (mismatched impedance), a reflection takes place. However, no reflections occur in one of two possible cases, the case of a fully absorbing object (matched impedance) and the case of open space with no intercepting objects.

Before we leave this section, let us finalize the expressions for the voltages and currents at any z-location on the line as well as at both the sending and receiving ends:

We start by rewriting Equation (2.41) in terms of the Γ's (reflection coefficients) associated with the source and load impedances,

$$\Gamma_L = \Gamma(\ell) = \frac{[Z_L - Z_o]}{[Z_L + Z_o]} \qquad (2.52)$$

$$\Gamma_s = \frac{[Z_S - Z_o]}{[Z_S + Z_o]} \qquad (2.53)$$

Note that Γ_s is different from $\Gamma(0)$. Γ_s is defined in Equation (2.53) as the reflection coefficient seen by the "–z" signal traveling back towards the source with the source impedance acting as the load to the Z_o line. On the other hand $\Gamma(0)$ is the reflection coefficient seen at the input port of the line by the "+z" wave, $\Gamma(z=0)=V^-/V^+$, Equation (2.50).

Now, back to Equation (2.41),

$$V^+ = \left\{\frac{V_S\, Z_o}{Z_S + Z_o}\right\}\left\{\frac{1}{1-\left(\frac{Z_S - Z_o}{Z_S + Z_o}\right)\left(\frac{Z_L - Z_o}{Z_L + Z_o}\right)e^{-2\gamma\ell}}\right\} = \left\{\frac{V_S\,[1-\Gamma_S]}{2}\right\}\left\{\frac{1}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\} = \left\{\frac{V_S}{2}\right\}\left\{\frac{1-\Gamma_S}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\} \qquad (2.54)$$

$$V^- = V^+\left(\frac{Z_L - Z_o}{Z_L + Z_o}\right)e^{-2\gamma\ell} = \left\{\frac{V_S}{2}\right\}\left\{\frac{1-\Gamma_S}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\}\left(\frac{Z_L - Z_o}{Z_L + Z_o}\right)e^{-2\gamma\ell} = \left\{\frac{V_S}{2}\right\}\left\{\frac{(1-\Gamma_S)\Gamma_L e^{-2\gamma\ell}}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\} \qquad (2.55)$$

Using Equations (2.44), (2.51), and (2.54), we can write the voltage as a function of position as follows:

$$V(z) = V^+ e^{-\gamma z}[1+\Gamma(z)] = \left\{\frac{V_s}{2}\right\}\left\{\frac{1-\Gamma_s}{1-\Gamma_s\,\Gamma_L e^{-2\gamma\ell}}\right\}e^{-\gamma z}\left[1+\Gamma_L e^{+2\gamma(z-\ell)}\right]$$

$$V(z) = \left\{\frac{V_S}{2}\right\}\left\{\frac{(1-\Gamma_S)\left[1+\Gamma_L e^{+2\gamma(z-\ell)}\right]}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\}e^{-\gamma z} \qquad (2.56)$$

Substituting z=0 and z=ℓ in this general expression, we can write the voltages at the sending and receiving ends as,

$$V(0) = \left\{\frac{V_S}{2}\right\}\left\{\frac{(1-\Gamma_S)\left(1+\Gamma_L e^{-2\gamma\ell}\right)}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\} \qquad (2.57)$$

$$V_L = V(\ell) = \left\{\frac{V_S}{2}\right\}\left\{\frac{(1-\Gamma_S)(1+\Gamma_L)\,e^{-\gamma\ell}}{1-\Gamma_S\,\Gamma_L e^{-2\gamma\ell}}\right\} \qquad (2.58)$$

Alternatively, in terms of impedances, we can write:

$$V(0) = V_s\left\{\frac{Z_o\{(Z_L+Z_o)+(Z_L-Z_o)e^{-2\gamma\ell}\}}{(Z_S+Z_o)(Z_L+Z_o)-(Z_S-Z_o)(Z_L-Z_o)e^{-2\gamma\ell}}\right\} \qquad (2.57a)$$

$$V_L = V_s\left\{\frac{2Z_L\,Z_o}{(Z_S+Z_o)(Z_L+Z_o)-(Z_S-Z_o)(Z_L-Z_o)e^{-2\gamma\ell}}\right\}e^{-\gamma\ell} \qquad (2.58a)$$

Two Special Cases: The Infinite Line and the Matched Load Line

Let us assume a TL with infinite length. As we just stated in the previous section, there will be no reflections from the load resulting in $\Gamma(\ell) = 0$, and hence Γ(z)=0 and $V^- = 0$.

Consequently, according to (2.18) and (2.23),

$$V(z) = V^+ e^{-\gamma z} \; and \; I(z) = \frac{1}{Z_o} \cdot \{V^+ e^{-\gamma z}\} \qquad (2.59)$$

which makes $Z(z) = Z_o$ at all z locations including z=0 (TL input impedance), $Z_{in} = Z(0) = Z_o$.

Solving the boundary condition at the source end,

$$V(0) = V^+ = V_s - Z_s \cdot I(0) = V_s - Z_s \cdot \frac{1}{Z_o}(V^+)$$

which yields:

$$V(0) = V^+ = V_s \frac{Z_o}{Z_s + Z_o} \; and \; I(0) = \frac{V^+}{Z_o} = V_s \frac{1}{Z_s + Z_o} \qquad (2.60)$$

Therefore, physically speaking, both an infinite line and a match terminated line exhibit matching at line locations including the line input (at z=0). Only waves traveling in +z direction can exist on the line.

Standing Waves and Standing Wave Ratio:

The combination of waves traveling in both directions on the same TL is likely to cause standing waves to form. To explain, let us start with the lossless line time-domain combined solution, Equation (2.34). By adding and subtracting the term $|V^-| \cos(\omega t - \beta z + \varphi_v^+)$ in the right hand side of the equation, we obtain:

$$v(z,t) = |V^+|\, cos(\omega t - \beta z + \varphi_v^+) + |V^-|\, cos(\omega t + \beta z + \varphi_v^-)$$
$$v(z,t) = \{[|V^+| - |V^-|]\, cos(\omega t - \beta z + \varphi_v^+)\} + \{|V^-|[cos(\omega t - \beta z + \varphi_v^+) + cos(\omega t + \beta z + \varphi_v^-)]\}$$
$$v(z,t) = \{[|V^+| - |V^-|]\, cos(\omega t - \beta z + \varphi_v^+)\} + \{2|V^-|[cos(\omega t + \varphi_1) \cdot cos(\beta z + \varphi_2)]\} \qquad (2.61)$$

Where φ_1 *and* φ_2 *replace the quantities* $\frac{\varphi_v^+ + \varphi_v^-}{2}$ *and* $\frac{\varphi_v^+ - \varphi_v^-}{2}$, *respectively*.

The first term of the resulting equation, $\{[|V^+| - |V^-|] \cos(\omega t - \beta z + \varphi_v^+)\}$, has the same traveling form as the positive-z traveling wave but with a reduced amplitude of $[|V^+| - |V^-|]$. On the other hand, the second term, $\{2|V^-|[\cos(\omega t + \varphi_1) \cdot \cos(\beta z + \varphi_2)]\}$ has an amplitude of $2|V^-|$ and describes a "standing wave" that does not travel in either direction on the line and t is characterized by stationary "peaks/maxima" and "valleys/minima". This can be seen mathematically by examining the expression and recognizing that the spatial dependence term $[\cos(\beta z + \varphi_2)]$ forces the term to zero at certain z locations irrespective of the time dependent term. This is demonstrated in Figure 2.13.

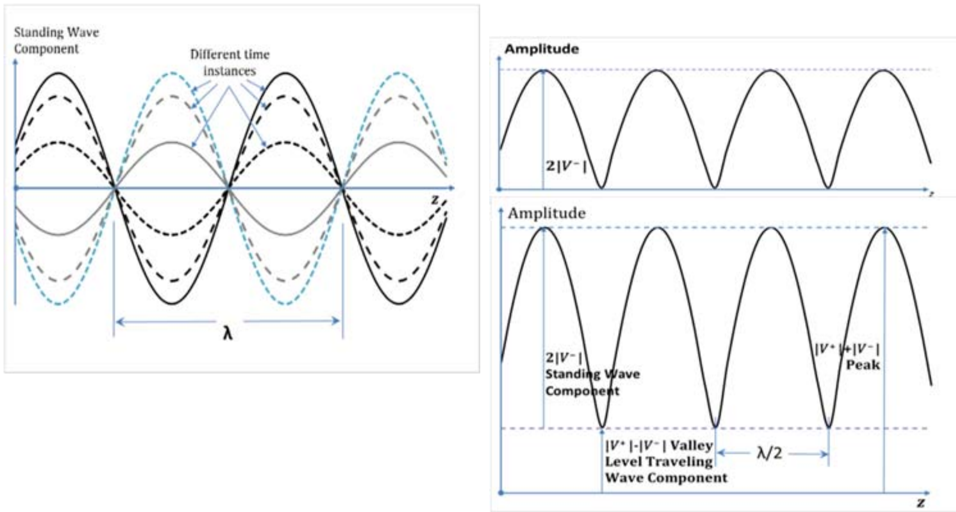


Figure 2.13

Hence, expression (2.61), as seen in the figure, is a mixture of a traveling wave component and a standing wave one. The amplitude of the standing wave component $2|V^-|$, is determined by the level of the reflected signal. In the absence of reflections, there will be no standing waves and we get pure traveling waves on the line. However, a 100% reflection where $|V^-| = |V^+|$ results in a pure "100%" standing waves on the TL since the traveling wave term $[|V^+| - |V^-|]$ reduces to zero.

Standing Wave Ratio:

An important indicator of the level of standing waves in a TL network is the standing wave ratio, SWR, the definition for which is the ratio of the peak (max) amplitude of the signal and its valley (min) value:

$$SWR = \frac{|V|_{max}}{|V|_{min}} = \frac{[|V^+|+|V^-|]}{[|V^+|-|V^-|]} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad (2.62)$$

SWR=1 for a pure traveling waves ($|\Gamma| = 0$) and SWR=∞ for a pure standing wave ($|\Gamma| = 1$).

Figure 2.14 is a graphical demonstration of the relationship between the SWR and $|\Gamma|$.

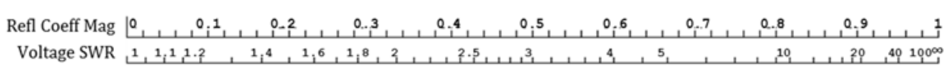


Figure 2.14

For lossy lines, the line attenuation causes the signals to decay down the TL path, resulting in the signal forms displayed in Figure 2.15.

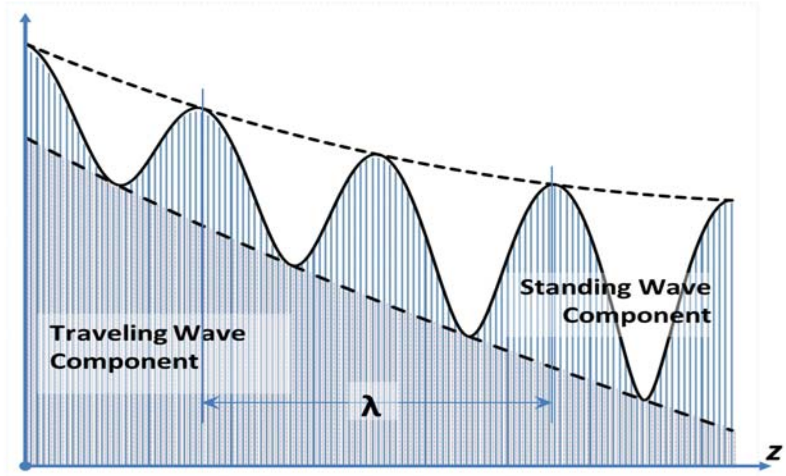


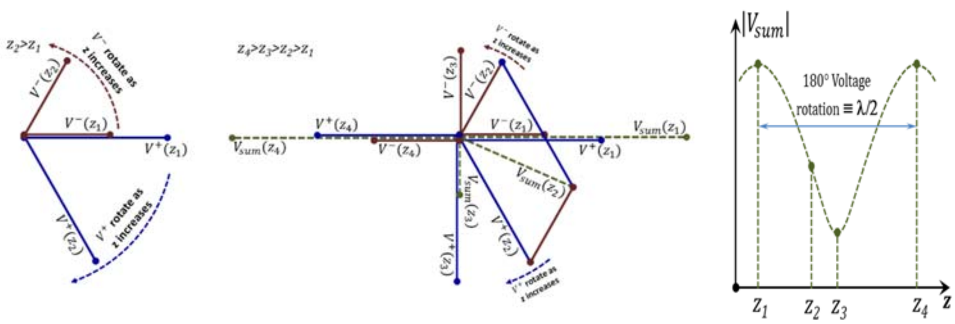
Figure 2.15

The figure shows the presence of a mixture of traveling and standing waves, just as in the lossless case. It is important to note here that since all the levels are changing, it is not possible to define a single value for the SWR. It is typical in such cases; to identify the location at which the SWR is given, i.e., the SWR of the load would be that of the nearest minimum and maximum to the TL load.

Another way of looking at the combination of the two traveling waves and their "interference" is to consider the corresponding phasor diagrams. For convenience, we will consider the case of a lossless TL only. In this case, the two phasors representing the two traveling waves are:

$$V^+e^{-j\beta z} \text{ and } V^-e^{+j\beta z}$$

Referring to Figure 2.16, we will start with a graph of the two phasors at some location z_1 where they line up in their phase. As we move from z_1 to z_2 (where $z_2>z_1$), the phase of the V+ phasor changes by $-\beta(z_2 - z_1)$ (decreases) while the phase of the V^- phasor changes by $+\beta(z_2 - z_1)$ (increases). Similarly, we monitor the phasors' rotation as we move to z_3 and z_4. The sum of the two phasors is displayed for all four positions in part (b) of the figure, and its magnitude is plotted vs distance in part (c) of the figure. We notice that the sum exhibits the same shape as that demonstrated earlier in Figure 2.13. We also notice that the peaks (maxima) occur when the two phasors line up with equal phases. The spacing between two consecutive peaks correspond to a 180 degree rotation which is equivalent to a distance if $\lambda/2$.



(a) ***(b)*** ***(c)***

Figure 2.16

Standing Waves and the Bounce Diagram:

In this section, we further our exploration of the physics of the multiple reflections / standing waves in mismatched TL networks. In this regard, we will consider the general case of impedance mismatches at both the source and load ends; hence, we can see that multiple wave reflections will occur on the line with waves bouncing back and forth between the (generally) mismatched TL ends. Figure 2.17 depicts the TL network and the expected "bounce" diagram as a result.

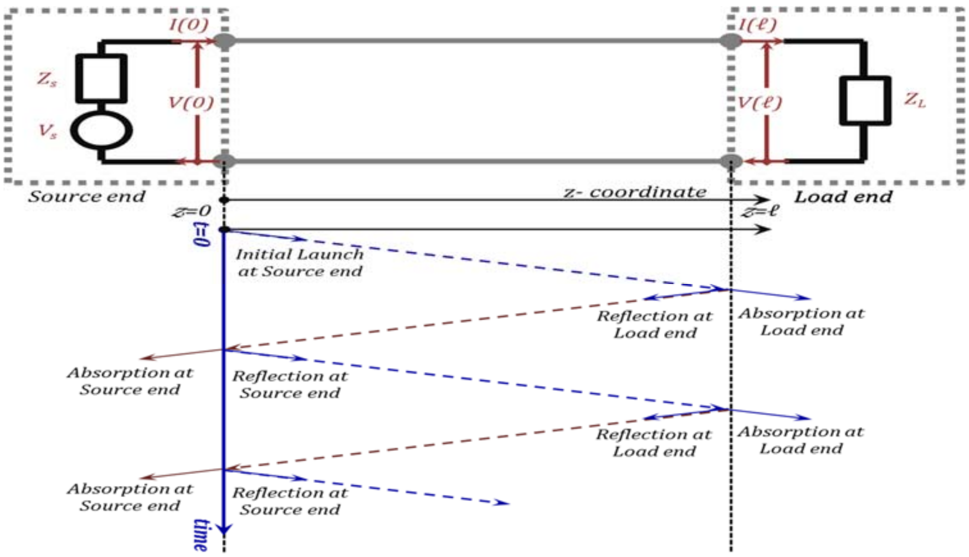


Figure 2.17

Using this bounce (reflection-transmission) diagram, let us now examine the physical meaning of the wave reflection and the corresponding reflection coefficient as defined in the frequency domain in Equation (2.50), $\Gamma(z) = \frac{V^- e^{+\gamma z}}{V^+ e^{-\gamma z}}$. We claim that the $V^- e^{+\gamma z}$ wave resulted from the reflections of the wave $V^+ e^{-\gamma z}$. Meanwhile, we do not see any discontinuities at a general location z that can cause reflections at that location. So, where did the $V^- e^{+\gamma z}$ come from? The answer lies in examining the transients build up before the steady state (harmonic) is reached.

In this regard, and based on the time-domain solution obtained earlier in Equation (2.29), a $v^+(z,t)$ wave, was initially launched at the sending end of the line. This wave traveled along the TL experiencing both attenuation and phase shift but no reflections until it met the first discontinuity at the load end. At the load "discontinuity", this wave suffered its first reflection triggering the backward traveling wave, $v^-(z,t)$. The reflected wave would then travel back to the source end experiencing only attenuation and phase change until it met the source end discontinuity. A source end reflection would then be in order and the bouncing back and forth continued. With each of these reflections, in general, some of the signal energy would be absorbed in the reflecting elements (the TL end termination impedances, Z_L *and* Z_s).

At steady state, there will continue to be a multiple reflections at both line ends resulting in a series of +z traveling waves and another series of –z traveling waves. The sum of terms of each of these two series correspond to the two terms of the steady-state harmonic solution, $V^- e^{+\gamma z}$ for the +z series and $V^+ e^{-\gamma z}$ for the –z series. (The proof of this statement will be carried out later in this chapter). The ratio between these two terms is what we defined as the reflection coefficient $\Gamma(z)$. So, as you can see, $\Gamma(z)$ does not physically exist as a localized phenomenon at the z location of the line; rather, it is a result of the comprehensive performance of the whole setup as seen by the line at the z location.

It is interesting to observe, that $\Gamma(z)$ has a localized physical meaning at $z = \ell$. At this location, and using Equation (2.49), $\Gamma_L = \Gamma(\ell) = \frac{[Z_L - Z_o]}{[Z_L + Z_o]}$ which is determined by the discontinuity mismatch at that location and is totally independent of the rest of system.

The Issues of Reflections and Standing Waves:

In this section, we investigate the issues with reflections and the resulting standing waves on transmission lines. To do that, we need to recognize that the purpose of a TL is to deliver the signal/power of the source/transmitter to the load/receiver end. Having said that, it is logical that a reflection from that end indicates the lack of a signal/power "full/adequate" delivery and that "some" of that signal/power was "denied/altered" and traveled back the "wrong" way. In terms of power, we can guess that more power could have been delivered to the load if none was reflected. Also, in terms of signal, the receiving end is not seeing the "one" signal that was intended but received multiple deliveries over time.

At this point, we need to demonstrate the concepts discussed above analytically, for both power and signal deliveries.

Power Delivery:

First, we need to understand the concept of power available from a source. To do so, let us assume a simple source load arrangement as shown in Figure 2.18. We will carry the analysis in the frequency domain for the convenience of avoiding differential equation formation and solutions. After all, we will be "swimming close to the shore" as we have done earlier and maintain close ties with the physics involved.

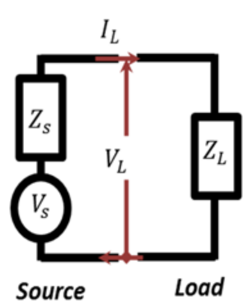


Figure 2.18

Now, let us evaluate the power delivered to the load in the simple source load circuit shown in the figure. The current in the load $I_L = V_s / (Z_s + Z_L)$ and hence, the power delivered to the load is given by:

$$P_L = |I_L|^2 \cdot Re[Z_L] \qquad (2.63)$$

Using the notations $Z_s = R_s + jX_s$ and $Z_L = R_L + jX_L$ where R_s, X_s, R_L, and X_L are the resistive and reactive parts of the source and load impedances, respectively, we can rewrite the expression in Equation (2.63) as follows:

$P_L = |V_s|^2 \frac{[R_L]}{[(R_s+R_L)^2+(X_s+X_L)^2]}$

This power is maximum when the following conditions are BOTH met:

$$X_s + X_L = 0 \; and \; R_s = R_L, which \; means \; Z_L = Z_s^* \qquad (2.64)$$

The reactance condition is simply telling us that the load reactance should null out (tune out) that of the source (or vice versa). So, we learn that in power transmission, since most of the loading (homes, factories, etc.) is inductive (motors, appliances, etc.), the load reactance is inductive and hence a capacitive source impedance is necessary to improve power delivery to these loads. This typically is referred to as capacitive power factor correction.

With tuned out reactances, the second condition signifies that the resistive components of the source and load must be matched to each other, $R_s = R_L$.

With both constraints met, the power delivered to the load is the maximum power possible to obtain from the source and hence the term power available, P_{av}

$$P_{av} = [P_L]_{max} = \frac{|V_s|^2}{4R_s} \qquad (2.65)$$

Now, how does this apply to our TL network? First, for the source to deliver its "maximum" available power, the impedance seen by the source ($Z_{in} = Z(0)$) is the loading impedance to the source. Hence, the requirement becomes:

$$Z_s^* = Z_{in} \xrightarrow{yields} \Gamma_s^* = \Gamma_{in} = \Gamma_L \cdot e^{-2\gamma\ell} \qquad (2.66)$$

Likewise, for the line to deliver maximum power to the load, the condition is: $\Gamma_L^* = \Gamma_s \cdot e^{-2\gamma\ell} \xrightarrow{yields} \Gamma_s^* = \Gamma_L \cdot [e^{2\gamma\ell}]^*$ *(2.67)*

Comparing the two conditions, we find out that both conditions can be simultaneously satisfied if and only if $e^{-2\gamma\ell} = [e^{2\gamma\ell}]^*$, which can only be true for lossless lines only, $\alpha = 0$.

Signal Delivery:

In this section, we will discuss the effect of multiple reflections (and hence the standing waves) on the quality (fidelity) of signal delivery. This is of prime concern in communication applications of TLs. For the receiving end not to get the "full" delivery of the transmitted signal and having to "reflect" part of it could be thought of as a reduction in the level of the received signal. The fact of the matter is that it can go beyond affecting the level to actually causing distortion to the received signal.

If we examine the expression for the receiving end signal, Equation (2.58), we can see that the relationship between VL and Vs includes, in general, complex coefficients. This implies that the signal arriving at the receiving end suffers both amplitude and phase changes. Typically, this is of no concern for single harmonic transmission applications. However, for communication applications, where a composite signal with components of multiple frequencies are propagated on the line, signal components of differing frequencies are expected to have different amplitude and phase modifiers.

The same can be said regarding digital communications where the information signal is coded in pulse format. These "time domain" pulses are rich in harmonics and thus contain a wide spectrum of frequencies. If the frequency components are to experience different "treatment" as they travel from the source to the load, the arrival composite of altered frequency components would result in "distorted" pulses, thus compromising the quality and fidelity of the signal delivery. This form of "pulse" distortion is called pulse spreading or "dispersion".

In addition to pulse spreading as discussed, multiple reflections result in multiple arrivals of time domain signals at the receiving end; see Figure 2.17. An example of this issue is demonstrated in Figure 2.19. In this example, a single pulse transmission on a line with multiple reflections is shown. The receiving end signal does not arrive to the load all at once; instead, the load receives multiple arrivals at different delay times. Therefore, the arrival at the receiving end is spread over an extended period due to the multiple occurrences in the time domain.

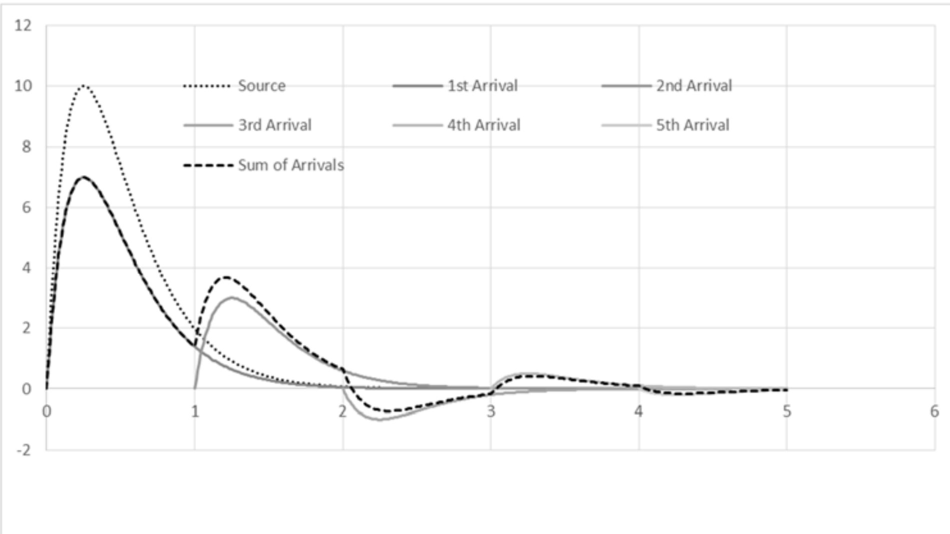


Figure 2.19

One obvious consequence is the serious limitation this form of dispersion causes to digital communication applications. The rate of sending information pulses has to be slowed down to allow adequate decay of the multiple reflections of one pulse before sending the next one.

Ideally, for communication purposes where the fidelity reception of the transmitter signal is very important, multiple reflections are undesirable. To eliminate multiple reflections, either Γ_s or Γ_L (or both) should be reduced to zero. Hence, according to Equation (2.58)),

$$\begin{cases} \Gamma_s = 0 \quad yields \quad V_L = \left\{\frac{V_s}{2}\right\}(1+\Gamma_L)\, e^{-\gamma\ell} \\ \Gamma_L = 0 \quad yields \quad V_L = \left\{\frac{V_s}{2}\right\}(1-\Gamma_s)\, e^{-\gamma\ell} \\ \quad \Gamma_s = \Gamma_L = 0 \quad yields \quad V_L = \left\{\frac{V_s}{2}\right\} e^{-\gamma\ell} \end{cases} \tag{2.68}$$

Furthermore, for signal fidelity, all frequency components of V_L should have the same propagation treatment as they travel from the source to the load. In other words, V_L/V_s should be frequency independent. Assuming $e^{-\gamma\ell}$ of the transmission line to produce the same signal delay for all frequencies, we add further constraints to the above requirements as follows:

$$\begin{cases} \Gamma_s = 0 \quad yields \quad V_L = \left\{\frac{V_S}{2}\right\}(1+\Gamma_L)\, e^{-\gamma\ell} - \quad Fidelity\ constraint: \Gamma_L\ should\ be\ real \\ \Gamma_L = 0 \quad yields \quad V_L = \left\{\frac{V_S}{2}\right\}(1-\Gamma_s)\, e^{-\gamma\ell} - \quad Fidelity\ constraint: \Gamma_s\ should\ be\ real \\ \qquad \Gamma_s = \Gamma_L = 0 \quad yields \quad V_L = \left\{\frac{V_S}{2}\right\}\, e^{-\gamma\ell} - \quad Fidelity\ constraint: none \end{cases} \tag{2.69}$$

The requirements for signal fidelity can be summarized as follows: either Γ_s=0 or Γ_L=0, while the non-zero one should be real.

Combined Power and Signal Delivery Constraints:

According to the conditions in Equations (2.66) and (2.67), if either Γ_s or Γ_L is zero, the other should be zero as well. Hence, to combine the two sets of requirements for maximum available power delivered to the load while maintaining signal fidelity, we need to have both Γ_s and Γ_L equal to zero, meanwhile, we should be using a lossless (or near lossless) TL.

In practical cases, we can design the source to have a matching internal impedance to the TL used and hence maintain Γ_s as zero. However, in many applications, we may not have control over the load impedance and consequently Γ_L. In such cases, a matching network becomes necessary to be implemented in order to enable the source-TL matched network to see an "equivalent" matched load. This will be discussed in detail in Addendum B: Impedance Matching.

Addendum A

Driving Point Impedance

TL Driving Point Impedance and Input Impedance:

It is obvious from the discussion above that the impedance relationships on a TL plays a dominant role in assessing its performance. For this reason, in the following section we will focus on developing some critical impedance relationships.

First, let us derive an expression for what we know as the "driving point impedance". This is the impedance seen by an observer at a given location on the line looking towards the load side, Figure 2.20.

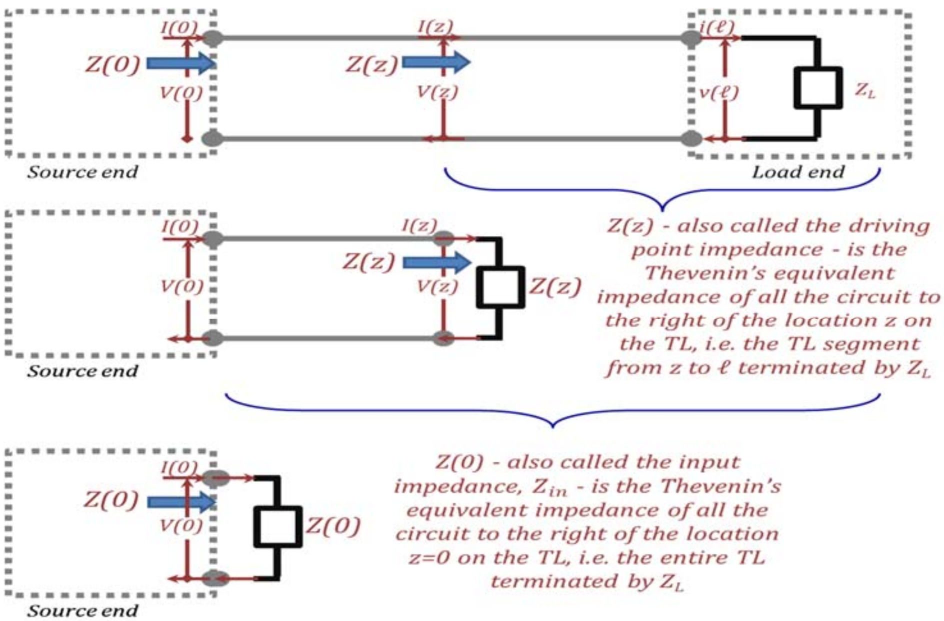


Figure 2.20

Using Equation (2.46) and (2.51) above,

$$Z(z) = \frac{V(z)}{I(z)} = Z_o \frac{[1+\Gamma(z)]}{[1-\Gamma(z)]} = Z_o \frac{[1+\Gamma(\ell)\cdot e^{-2\gamma d}]}{[1-\Gamma(\ell)\cdot e^{-2\gamma d}]} = Z_o \frac{\left[1+\frac{[Z_L-Z_o]}{[Z_L+Z_o]}\cdot e^{-2\gamma d}\right]}{\left[1-\frac{[Z_L-Z_o]}{[Z_L+Z_o]}\cdot e^{-2\gamma d}\right]}$$

$$Z(z) = Z_o \frac{\left[[Z_L+Z_o]+[Z_L-Z_o]\cdot e^{-2\gamma d}\right]}{\left[[Z_L+Z_o]-[Z_L-Z_o]\cdot e^{-2\gamma d}\right]} = Z_o \frac{\left[[Z_L+Z_o]\cdot e^{+\gamma d}+[Z_L-Z_o]\cdot e^{-\gamma d}\right]}{\left[[Z_L+Z_o]\cdot e^{+\gamma d}-[Z_L-Z_o]\cdot e^{-\gamma d}\right]}$$

$$Z(d) = Z(z) = Z_o \frac{[2Z_L\cdot cosh(\gamma d)+2Z_o\cdot sinh(\gamma d)]}{[2Z_L\cdot sinh(\gamma d)+2Z_o\cdot cosh(\gamma d)]} = Z_o \frac{[Z_L+Z_o\cdot tanh(\gamma d)]}{[Z_o+Z_L\cdot tanh(\gamma d)]} \qquad (2.70)$$

Hence,

$$Z_{in} = Z(z=0) = Z_o \frac{[Z_L+Z_o\cdot tanh(\gamma\ell)]}{[Z_o+Z_L\cdot tanh(\gamma\ell)]} \qquad (2.71)$$

To gain some insight into this expression, let take the lossless transmission line case where

$\gamma = j\beta$ *and* $Z_o = R_o$,

$Z(d) = R_o \frac{[Z_L + R_o \cdot j\ tan(\beta d)]}{[R_o + Z_L \cdot j\ tan(\beta d)]}$, (2.72)

and since $\tan[\beta(d \pm n\lambda/2)] = \tan\left(\frac{2\pi d}{\lambda} \pm \pi\right) = \tan(2\pi d/\lambda) = \tan(\beta d)$, then

$Z(d \pm n\lambda/2) = Z(d)\ or\ Z(z \pm n\lambda/2) = Z(z)$ (2.73)

In other words, the impedance values repeat on a lossless line every half-wavelength while, as we recall, the voltage waves currents repeated every full wavelength.

Furthermore, since $\tan\ [\beta(d \pm n\lambda/4)] = \tan\left(\frac{2\pi d}{\lambda} \pm \pi/2\right) = -\{\tan(\beta d)\}^{-1}$, then

$Z(d \pm n\lambda/4) = R_o \frac{[Z_L + R_o \cdot \{j\ tan(\beta d)\}^{-1}]}{[R_o + Z_L \cdot \{j\ tan(\beta d)\}^{-1}]} = R_o \frac{[R_o + Z_L \cdot j\ tan(\beta d)]}{[Z_L + R_o \cdot j\ tan(\beta d)]} = R_o^2 / Z(d)$ (2.74)

Therefore, every quarter-wavelength the lossless TL driving point impedance inverses its nature.

Some Special Cases:

For a lossless line (almost true for low loss lines as well), we can summarize some features derived from Equation (2.72) in the Table (2.2):

Table 2.2

$Z(d)$	*Z(z), all values*	$Z(d \pm n\lambda/2)$	$Z(d \pm n\lambda/4)$
R_o *(Matched)*	R_o *(Matched)*	R_o *(Matched)*	R_o *(Matched)*
0 (Short Circuit)	*Reactive values:* $-\infty \le X(z) \le +\infty$	*Same (=*Z_d*)*	*∞ (Open Circuit)*
∞ (Open Circuit)	*Reactive values:* $-\infty \le X(z) \le +\infty$	*Same (=*Z_d*)*	*0 (Short Circuit)*
$Z(d)$ *has positive reactance (inductive)*	*Reactive values:* $-\infty \le X(z) \le +\infty$	*Same (=*Z_d*)*	*negative reactance (capacitive)*
$Z(d)$ *has negative reactance (capacitive)*	*Reactive values:* $-\infty \le X(z) \le +\infty$	*Same (=*Z_d*)*	*positive reactance (inductive)*
$\lvert Z(d)\rvert < R_o$		*Same (=*Z_d*)*	$\lvert Z(d)\rvert > R_o$
$\lvert Z(d)\rvert > R_o$		*Same (=*Z_d*)*	$\lvert Z(d)\rvert < R_o$

It is interesting to know that some of features cited in the above table have useful applications. One typical application is the use of a quarter-wavelength TL section to act as an impedance transformer (adapter).

$Z_{in}|_{\lambda/4} = R_o^2 / Z_L$ (2.75)

Examples: A quarter-wavelength TL section terminated by a capacitor works as an inductor for use in high frequency devices. This is a highly preferred approach since wound coils do not offer the desired performance at radio and microwave frequencies. We find a similar approach in using a short-circuited quarter-wavelength TL section to act as an open circuit at high frequencies.

Another typical application is to use a quarter-wavelength TL section with the right value of R_o to provide impedance matching between differing source and load resistive impedances.

$R_o^2 = R_{in}|_{\lambda/4} \cdot R_L = R_s \cdot R_L$, e.g. a 150 Ω quarter-wavelength TL can match a 75 Ω source to a 300 Ω load.

Addendum B

Impedance Matching

How to Achieve Matching:

Referring to Figures 2.21 – 2.24, matching the load impedance to the source impedance implies making the projected load impedance at the source end equal to that of the source. In other words, making the input impedance of the load "network" equal the source impedance. It is worthy to mention here that although the discussion in this section is focused on matching the load to the source, the opposite is possible and doable through the same approaches presented here. It is also important to cite that in typical practical cases, the source impedance is pure resistive, and hence that is what we will confine our discussion in this section to.

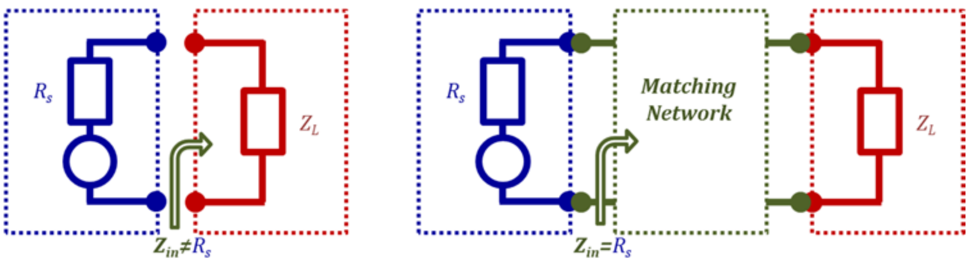


Figure 2.21

To bring the "projected" load impedance as viewed at the source port to the source impedance (resistance) value, a matching network is needed to act as an impedance transformer between the two values.

One may argue that a simple series or shunt impedance matching network would achieve the desired result, Figure 2.22. For example, if the load impedance (resistance) level is lower than that of the source then add a series impedance in the amount of $Z_{series}=Z_s-Z_L$, thus making $Z_{in}=Z_s$. On the other hand, if the load impedance (resistance) level is higher than that of the source, then add the matching network impedance in shunt, and hence $Z_{in}=Z_L//Z_{shunt}$. The problem with these configurations is that either Z_{series} or Z_{shunt} are likely to contain resistive components and hence introducing loss elements to the circuit that would degrade its performance.

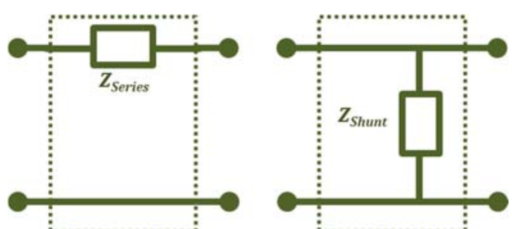


Figure 2.22

As the networks of Figure 2.22 prove inadequate, some of the typically used matching network forms are shown in Figures 2.23, 24, and 25.

L-PI-T Matching Networks:

Figure 2.23 shows three matching network configurations: the "L-section", the "π-network", and the "T-network".

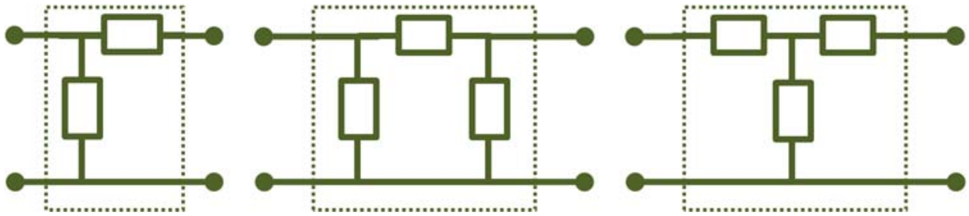


Figure 2.22

The literature has many references regarding the design approaches of such networks as well as comparisons of their performances. Depending on the network configuration, the design may be optimized in terms of bandwidth or other possible performance parameters.

For optimum performance, all matching network components must not add any loss to the circuit (or at least be "low loss" components). In other words, the network elements need to be high quality "high-Q" reactances. Depending on the operating frequency of the circuit, discrete components cannot be used for lumped circuit elements sought in these networks as they may cause performance issues. If the size of the discrete components is not much smaller than the operating wavelength, as discussed earlier in Table (2.1), the component must be treated as a distributed "transmission line" device instead of a lumped element. Moreover, the physical devices for these elements will not be the pure reactive components sought in the design, as they will most certainly contain other parasitics.

Stub Matching:

At high frequencies, lossless transmission lines (or practically, low loss lines) are used instead of discrete components. Typical transmission line matching networks are shown in Figures 2.24-25. The design principle of the networks shown is to use a transmission line on the load side with the proper line parameters that would project a resistive value at its input port so that it would match the resistive values of the source impedance. Tuning reactances can then be added to achieve the desired full impedance match.

To further explain, assume we are working with the top configuration of Figure 2.24. We will need to choose TL parameters $R_{o,ms}$, β_{ms}, and d_{ms} such that the real part of ($Z_d = Z_{in,ms}$) is equal to R_s. Using Equation (2.72), we can write:

$$Real\ part\ of \left\{ Z_d = R_{o,ms} \frac{[Z_L + R_{o,ms} \cdot j\ tan(\beta_{ms} d_{ms})]}{[R_{o,ms} + Z_L \cdot j\ tan(\beta_{ms} d_{ms})]} \right\} = R_s \qquad (2.76)$$

The added series reactance $\pm\ j\ X_{series}$ is to tune out the imaginary part (reactance) of Z_d, and hence

$$\pm j X_{series} + Imaginary\ part\ of \left\{ Z_d = R_{o,ms} \frac{[Z_L + R_{o,ms} \cdot j\ tan(\beta_{ms} d_{ms})]}{[R_{o,ms} + Z_L \cdot j\ tan(\beta_{ms} d_{ms})]} \right\} = 0 \qquad (2.77)$$

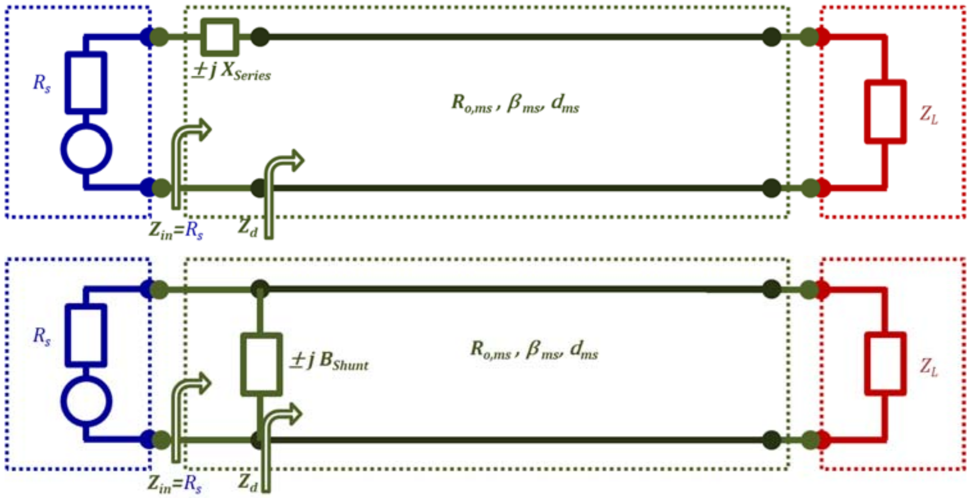


Figure 2.24

Similarly, the equations for the bottom configuration with the shunt reactance can be written in terms of the conductance and susceptance terms instead of resistance and reactance terms; hence,

$$Real\ part\ of\left\{Y_d=\left[R_{o,ms}\frac{[Z_L+R_{o,ms}\cdot j\tan(\beta_{ms}d_{ms})]}{[R_{o,ms}+Z_L\cdot j\tan(\beta_{ms}d_{ms})]}\right]^{-1}\right\}=[R_s]^{-1} \qquad (2.78)$$

and an added shunt susceptance $\pm jB_{shunt}$ is to tune out the imaginary part (susceptance) of Y_d; therefore,

$$\pm jB_{shunt}+Imaginary\ part\ of\left\{Y_d=\left[R_{o,ms}\frac{[Z_L+R_{o,ms}\cdot j\tan(\beta_{ms}d_{ms})]}{[R_{o,ms}+Z_L\cdot j\tan(\beta_{ms}d_{ms})]}\right]^{-1}\right\}=0 \quad (2.79)$$

To realize the series reactance or shunt susceptance needed in the matching networks of Figure 2.24, series and parallel transmission line "stubs" as shown in Figure 2.25 are utilized. The idea is that an open- or short-circuited transmission line at one of its ends presents a pure reactive/susceptive input impedance at the other end, refer back to Table (2.2). Typically, a short-circuited stub is used instead of an open-circuited one to avoid open-end radiation and associated losses at the open end of the line.

For a short-circuited stub, the corresponding parameters are to be chosen so that (using Equation (2.72) with $Z_L=0$)

$$\pm jX_{series}=j\,R_{o,st}\cdot\tan(\beta_{st}d_{st})\ \textit{for the series stub, and} \qquad (2.80)$$
$$\pm jB_{shunt}=\left[j\,R_{o,st}\cdot\tan(\beta_{st}d_{st})\right]^{-1} \qquad (2.81)$$

If an open stub is preferred, the equations for the stub reactances can be obtained for Equation (2.56) by setting Z_L to ∞ (or an estimate to its physical value). The results would be

$$\pm jX_{series}=-\,j\,R_{o,st}\cdot\cot(\beta_{st}d_{st})\ \textit{for the series stub, and} \qquad (2.82)$$
$$\pm jB_{shunt}=\left[-\,j\,R_{o,st}\cdot\cot(\beta_{st}d_{st})\right]^{-1} \qquad (2.83)$$

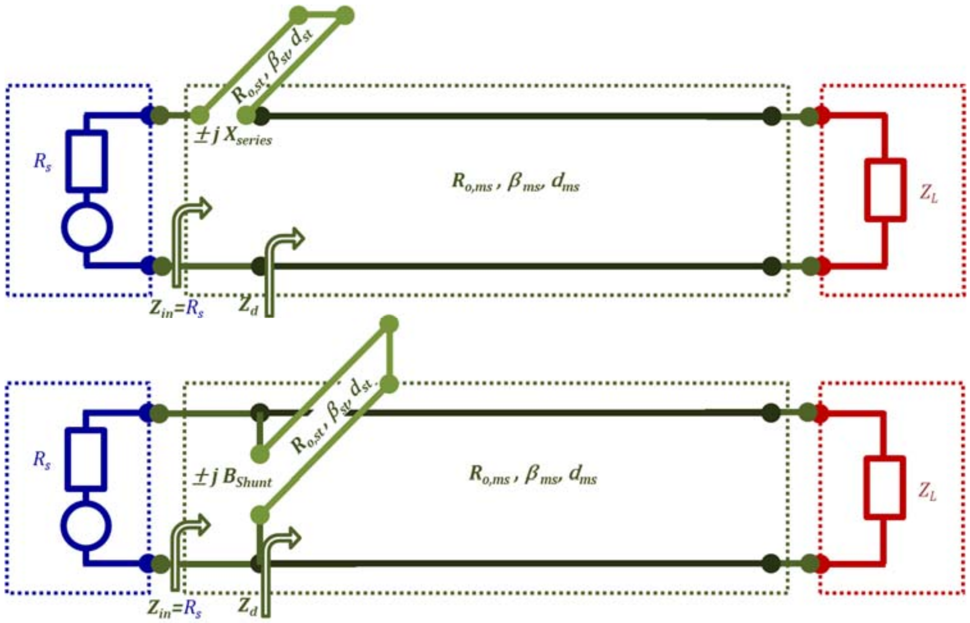


Figure 2.25

Later in this chapter, we will introduce the famous transmission line graphical technique known as the Smith Chart. This technique will be demonstrated to not only produce relatively simple computational solutions to transmission line problems, but also provide physical insight into the solution and the different parameters involved. As an application to the Smith chart, we will do the transmission line matching networks discussed above.

The Quarter Wave Transformer as a Matching Network:

In some special cases, a quarter wave transmission line section can be used to provide the desired matching between the source and the load.

Using Equation (2.72) again and setting d=λ/4, we can write:

$$Z_{in} = \frac{{R_o}^2}{Z_L} \qquad (2.84)$$

The resulting projected impedance Z_{in} is to be equated to R_s for matching with the source impedance (resistance); hence,

$$R_s = \frac{{R_o}^2}{Z_L}$$

which implies that ZL should be real. In other words, a lossless/low-loss quarter wave transformer is good for matching real (resistive) load impedances to resistive source impedances provided that we choose the characteristic impedance of the quarter wave section to be

$$R_o = \sqrt{R_s \cdot R_L} \qquad (2.85)$$

Figure 2.26 demonstrates the use of a λ/4 section for matching purposes.

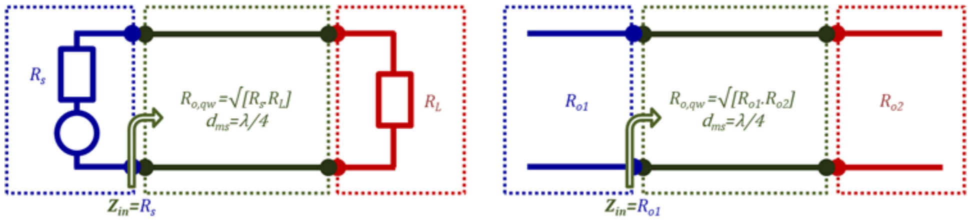


Figure 2.26

The Half Wave Transformer as a Matching Network:

Now, we wonder, what can a half wavelength section be used for? Can it be used for matching as well? The answer starts by using Equation (2.72) while setting d=λ/2, which yields:

$$Z_{in} = Z_L \qquad (2.86)$$

which is independent of R_o (the characteristic impedance of the half wave section). This means that the load must be already matched to the source, which may lead us to think that the $\lambda/2$ section has no practical use in matching problems.

Well, let us consider the case where a matched source and load pair must be connected, for some practical reason, by a TL whose characteristic impedance is different. In such case, a λ/2 section of this connecting line would be the ideal length, as it would maintain the existing matching between the source and the load. Such cases exist in real life where connecting transmission line sections must take a non-uniform shape such as a bent or a twist, etc. These sections need to be designed to have a half wavelengths in order to maintain matching, see Figure 2.27.

Figure 2.27

Addendum C

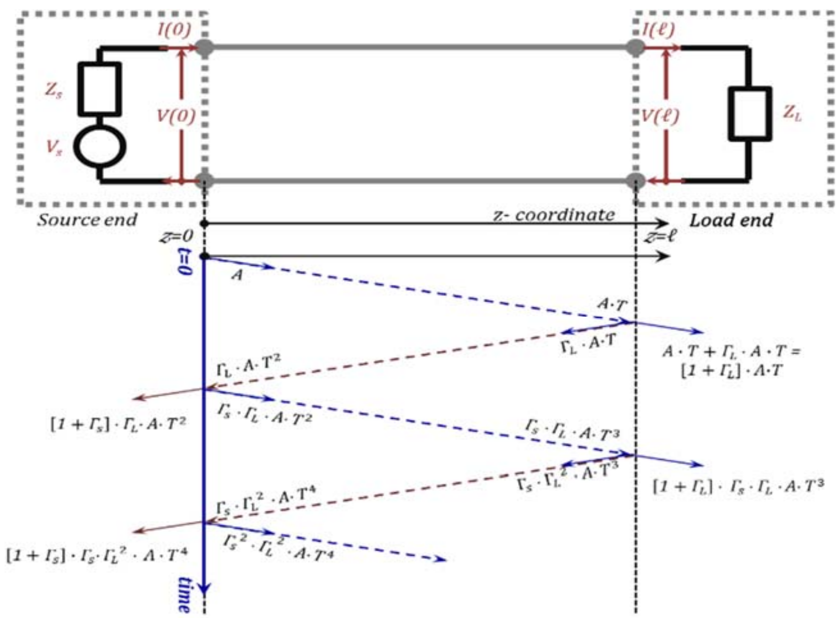
The Bounce Diagram in Frequency Domain

Bounce Diagrams in the Frequency Domain:

The title of this section may sound provocative to some. We talk about the evolution of the signals on the TL, i.e. transients, not in the time domain, but in the frequency domain. This is what we cited in Chapter I when we said you might find people communicating in French in the state of England. There must be a good reason for doing so, and the following will demonstrate the idea.

So, we start by recognizing that an initial launch was let by the source. The initial voltage wave corresponding to this launch is labeled "A" in Figure 2.28. This phasor voltage appears across the input terminals of the TL. Referring to Figure 2.29, the launched signal at t=0 sees only the input side of the TL and does not see the TL discontinuity of the load side till the signal arrives there after a delay of ℓ/c_{ph}. No reflections appear at this moment and hence only $V^+e^{-\gamma z} = V^+$ exists. The corresponding current would be $[V^+e^{-\gamma z}]/Z_o = V^+/Z_o$, which means that the input impedance of the TL at that instant (t=0) is equal to Z_o. The initial launch voltage phasor, A, would then be given by a voltage division of the source voltage V_s between the source internal impedance Z_s and the TL input impedance Z_o.

$$A = V(0)|_{t=0} = V_s \cdot \frac{Z_o}{[Z_s + Z_o]} \qquad (2.87)$$



$$A = V(0)|_{t=0} \quad \text{--} \quad A = V_s \cdot \frac{Z_o}{[Z_s + Z_o]}$$

A is the initial launch voltage wave phasor. Initially, this launch does not "see" the end of the line, and hence, the line appears to have an input impedance of Z_o similar to that of an infinite line $T = e^{-\gamma\ell}$ -- $\Gamma_L = \Gamma(\ell) = \frac{[Z_L - Z_o]}{[Z_L + Z_o]}$ -- $\Gamma_s = \frac{[Z_s - Z_o]}{[Z_s + Z_o]}$

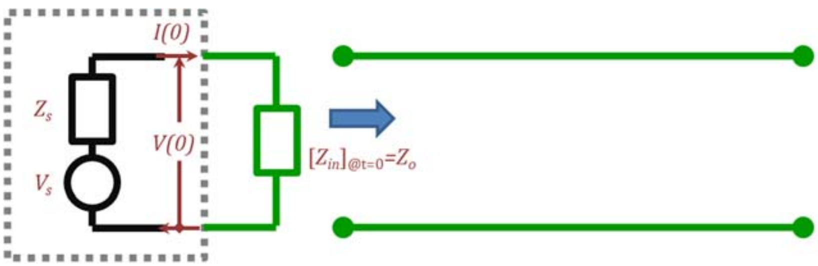
Figure 2.28

In the following, we need to refer to both Figures 2.28 and 2.30. The difference between the two figures is that 2.28 shows the multiple reflections at both line ends while 2.30 adds the line signals at an interim location (z). As demonstrated in both figures, the initial launch (A) travels down the TL experiencing modifications in the form of attenuation, $e^{-\alpha z}$, and phase change, $e^{-j\beta z}$. So, at location z on the line the modified initial launch will take the form of the phasor $A \cdot e^{-\gamma z}$ which we will label at $A \cdot T_z$, and by the end of the line at $z = \ell$, the phasor will take the form $[V^{+}e^{-\gamma\ell}]_{@t=\ell/c} = A \cdot e^{-\gamma\ell} = A \cdot T$. This positive traveling wave will bounce off the Z_L discontinuity causing a negative z-traveling wave at $z = \ell$, viz.,

$$[V^{-}e^{+\gamma\ell}]_{@t=\ell/c} = \Gamma_L \cdot \ [V^{+}e^{-\gamma\ell}]_{@t=\frac{\ell}{c}} = \Gamma_L \cdot A \cdot T$$

The total voltage phasor at that location and at that instant would be

$$V(\ell) = \ [V^{+}e^{-\gamma\ell}]_{@t=\ell/c} + \ [V^{-}e^{+\gamma\ell}]_{@t=\ell/c} = A \cdot T + \Gamma_L \cdot A \cdot T = [1 + \Gamma_L] \cdot A \cdot T$$



$$V(0)|_{t=0} = V_S \cdot \frac{Z_O}{[Z_S + Z_O]}$$

A is the initial launch voltage wave. Initially, this launch does not "see" the end of the line, and hence, the line appears to have an input impedance of Z_o

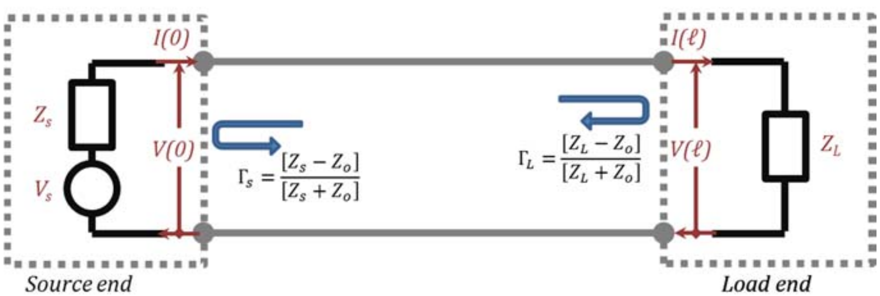
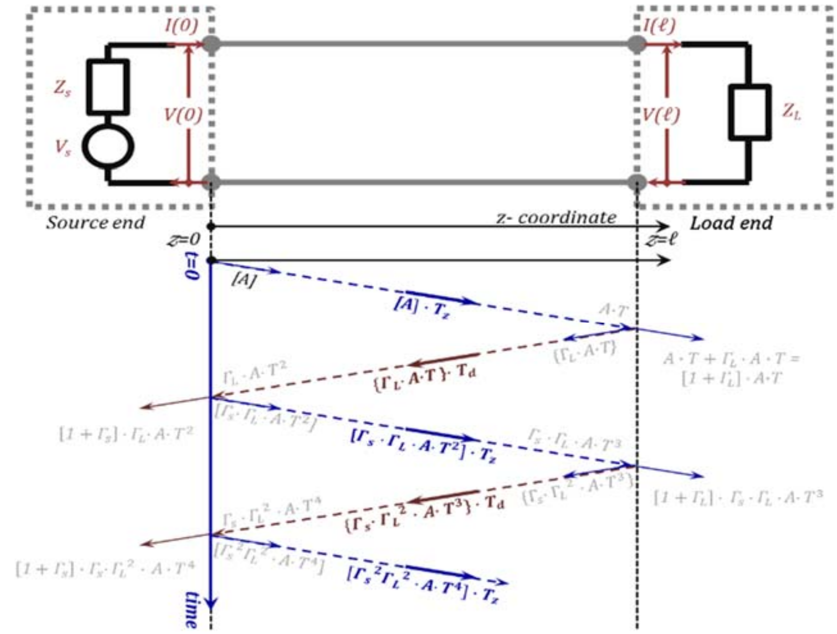


Figure 2.29

This is the voltage phasor across the load termination, and hence, we may call it the transmitted voltage phasor (at that location and at that instant). This component will be "absorbed" by the load.



$$T=e^{-\gamma\ell}\ \text{-}\ T_z=e^{-\gamma \mathrm{z}}\ \text{-}\ T_d=e^{-\gamma \mathrm{d}}\ \text{-}\ d=\ell-\mathrm{z}$$

Figure 2.30

Next, we examine the reflected phasor for the load side. Starting with the $[V^{-}e^{+\gamma\ell}]_{@t=\ell/\mathrm{c}}=\{\Gamma_L\cdot A\cdot T\}$, this phasor changes through the multiplier $e^{-\gamma\cdot distance\ traveled}$, so, at location z, the distance traveled would be $d=\ell-\mathrm{z}$ and at the source end the distance traveled would be ℓ. Hence the –z traveling wave will have its phasor expressions as $e^{-\gamma\cdot d}\cdot\{\Gamma_L\cdot A\cdot T\}=T_d\cdot\{\Gamma_L\cdot A\cdot T\}$ and $e^{-\gamma\cdot\ell}\cdot\{\Gamma_L\cdot A\cdot T\}=T\cdot\{\Gamma_L\cdot A\cdot T\}$=$\Gamma_L\cdot A\cdot T^2$. This –z wave arrives at the source end at $t=2\ell/\mathrm{c}$. When it does, it will be faced with the discontinuity of the source impedance, causing a reflection and a transmission (similar to what happened at the load end). The reflection coefficient facing the –z propagation at the source end will be given by $\Gamma_s=\frac{[Z_s-Z_o]}{[Z_s+Z_o]}$. This makes the reflected and transmitted phasors $[\Gamma_s\cdot\Gamma_L\cdot\ A\cdot T^2]$ and $[1+\Gamma_s]\cdot\Gamma_L\cdot\ A\cdot T^2$, respectively. Following the same approach, we can construct the following table, Table 2.3, for the phasor bounce diagram:

$$A=V_s\cdot\frac{Z_o}{[Z_s+Z_o]}=V_s\cdot\frac{[1-\Gamma_s]}{2} \qquad (2.88)$$

Table 2.3

@ Source End			*@ location z,* ($\mathbf{d}=\boldsymbol{\ell}-\mathbf{z}$)		*@ Load End*		
		+z		*+z*		*+z*	
	-z		*-z*		*-z*		*To* $\boldsymbol{Z_L}$
To $\boldsymbol{Z_s}$							
		[A]		$[A]\,T_z$		AT	
$\Gamma_L AT^2$		$\{\boldsymbol{\Gamma_L AT}\}\boldsymbol{T_d}$		$\{\Gamma_L AT\}$		$[1+\Gamma_L]AT$	
$[1+\Gamma_s]\Gamma_L AT^2$		$[\Gamma_s\Gamma_L AT^2]$		$[\boldsymbol{\Gamma_s\Gamma_L AT^2}]\,T_z$		$\Gamma_s\Gamma_L AT^3$	
$\Gamma_s\Gamma_L^2 AT^4$		$\{\boldsymbol{\Gamma_s\Gamma_L^2 AT^3}\}T_d$		$\{\Gamma_s\Gamma_L^2 AT^3\}$		$[1+\Gamma_L]\Gamma_s\Gamma_L AT^3$	
$[1+\Gamma_s]\Gamma_s\Gamma_L^2 AT^4$		$[\Gamma_s^2\Gamma_L^2 AT^4]$		$[\boldsymbol{\Gamma_s^2\Gamma_L^2 AT^4}]\,T_z$		$\Gamma_s^2\Gamma_L^2 AT^5$	
$\Gamma_s^2\Gamma_L^3 AT^6$		$\{\boldsymbol{\Gamma_s^2\Gamma_L^3 AT^5}\}\,T_d$		$\{\Gamma_s^2\Gamma_L^3 AT^5\}$		$[1+\Gamma_L]\Gamma_s^2\Gamma_L^2 AT^5$	

$[1+\Gamma_s]\Gamma_s^2\Gamma_L^3AT^6$	$[\Gamma_s^3\Gamma_L^3AT^6]$	$[\boldsymbol{\Gamma_s^3\Gamma_L^3AT^6}]T_z$	

		Phasor Series	*Expression Sum*
@ Source	To $\mathbf{Z_s}$	$[1+\Gamma_s]\Gamma_LAT^2+[1+\Gamma_s]\Gamma_s\Gamma_L^2AT^4+[1+\Gamma_s]\Gamma_s^2\Gamma_L^3AT^6+\cdots$	$\frac{[1+\Gamma_s]\Gamma_LT^2}{[1-\Gamma_s\Gamma_LT^2]}A$
	-z	$\Gamma_LAT^2+\Gamma_s\Gamma_L^2AT^4+\Gamma_s^2\Gamma_L^3AT^6+\cdots$	$\frac{\Gamma_LT^2}{[1-\Gamma_s\Gamma_LT^2]}A$
	+z	$A+\Gamma_s\Gamma_LAT^2+\Gamma_s^2\Gamma_L^2AT^4+[\Gamma_s^3\Gamma_L^3AT^6]+\cdots$	$\frac{A}{[1-\Gamma_s\Gamma_LT^2]}$
@ z	-z	$\boldsymbol{V^-(z)}=\{\boldsymbol{\Gamma_L}AT\}T_d+\{\boldsymbol{\Gamma_s\Gamma_L^2AT^3}\}T_d+\{\boldsymbol{\Gamma_s^2\Gamma_L^3AT^5}\}T_d+\cdots$	$\frac{\boldsymbol{\Gamma_LTT_d}}{[\mathbf{1}-\boldsymbol{\Gamma_s\Gamma_LT^2}]}\boldsymbol{A}$
	+z	$\boldsymbol{V^+(z)}=[A]T_z+[\boldsymbol{\Gamma_s\Gamma_L}A\boldsymbol{T^2}]T_z+[\boldsymbol{\Gamma_s^2\Gamma_L^2}A\boldsymbol{T^4}]T_z+\cdots$	$\frac{\boldsymbol{T_z}}{[\mathbf{1}-\boldsymbol{\Gamma_s\Gamma_LT^2}]}\boldsymbol{A}$
@ Load	-z	$\{\Gamma_LAT\}+\{\Gamma_s\Gamma_L^2AT^3\}+\{\Gamma_s^2\Gamma_L^3AT^5\}+\cdots$	$\frac{\Gamma_LT}{[1-\Gamma_s\Gamma_LT^2]}A$
	+z	$AT+\Gamma_s\Gamma_LAT^3+\Gamma_s^2\Gamma_L^2AT^5+\cdots$	$\frac{AT}{[1-\Gamma_s\Gamma_LT^2]}A$
	To $\mathbf{Z_L}$	$[1+\Gamma_L]AT+[1+\Gamma_L]\Gamma_s\Gamma_LAT^3+[1+\Gamma_L]\Gamma_s^2\Gamma_L^2AT^5+\cdots$	$\frac{[1+\Gamma_L]T}{[1-\Gamma_s\Gamma_LT^2]}A$

The expressions in the table can be used to construct answers for the voltage phasors as functions of position, i.e.:

At the source end:

$$V^+=\frac{A}{[1-\Gamma_s\Gamma_LT^2]}=\left\{V_s\cdot\frac{[1-\Gamma_s]}{2}\cdot\frac{1}{[1-\Gamma_s\cdot\Gamma_L\cdot T^2]}\right\} \tag{2.89}$$

$$V^-=\frac{\Gamma_L\,T^2A}{[1-\Gamma_s\Gamma_LT^2]}=\left\{V_s\cdot\frac{[1-\Gamma_s]}{2}\cdot\frac{1}{[1-\Gamma_s\cdot\Gamma_L\cdot T^2]}\cdot\Gamma_L\,T^2\right\}=V^+\,\Gamma_L\,T^2 \tag{2.90}$$

$$V(0)=\{V^+\}+\{V^-\} \tag{2.91}$$

At location z:

$$V^+(z)=\frac{T_z\,A}{[1-\Gamma_s\Gamma_LT^2]}=\left\{V_s\cdot\frac{[1-\Gamma_s]\,T_z}{2\cdot[1-\Gamma_s\cdot\Gamma_L\cdot T^2]}\right\}=\{V^+\}\cdot T_z\;=\{V^+\}\cdot e^{-\gamma\cdot z} \tag{2.92}$$

$$V^-(z)=\frac{\Gamma_L\,T\,T_d\,A}{[1-\Gamma_s\Gamma_LT^2]}=\left\{V_s\cdot\frac{[1-\Gamma_s]\,\Gamma_L\,T\,T_d}{2\cdot[1-\Gamma_s\cdot\Gamma_L\cdot T^2]}\right\}=\{V^-\}/T_z\;=\{V^-\}\cdot e^{+\gamma\cdot z} \tag{2.93}$$

$$V(z)=\{V^+\}\cdot e^{-\gamma\cdot z}+\{V^-\}\cdot e^{+\gamma\cdot z} \tag{2.94}$$

At the load:

$$V(\ell)=\{V^+\}\cdot e^{-\gamma\cdot\ell}+\{V^-\}\cdot e^{+\gamma\cdot\ell} \tag{2.95}$$

It is important to mention that the above obtained expressions for $\{\mathbf{V}^+\}$ and $\{\mathbf{V}^-\}$ are in agreement with those obtained earlier in Equations (2.41) and (2.42). This can be shown upon the substitution of $\Gamma_s=\frac{[Z_s-Z_o]}{[Z_s+Z_o]}$, $\Gamma_L=\frac{[Z_L-Z_o]}{[Z_L+Z_o]}$, and $\frac{V_s\,Z_o}{Z_s+Z_o}=V_s\cdot\frac{[1-\Gamma_s]}{2}$.

Addendum D

The Time Domain Bounce Diagram

The Time Domain Bounce Diagram:

When we discussed the bounce diagram in the frequency domain, we were thinking of harmonic signals traveling back and forth on the line (in the steady state). Although we were discussing a steady state, we used a “transient” approach to construct the steady state answer. To do the same exact thing in the time domain, i.e., use the transient approach to construct the time domain steady state answer, we will be limit the discussion to the lossless line case. This is because the time domain analysis of lossy lines gets beyond our mathematical capabilities and our interest at this point.

Two main advantages to working the bounce diagram in the time domain are the following: First, is the ability to work out transients of non-harmonic signals in simple expressions (only for lossless lines). The second reason is related to practical application in time domain reflectometry (including practical transmission media, which are not lossless).

In time domain reflectometry, a pulse is launched in a transmission medium and its reflections are monitored. The delay of the reflected signals is used to determine the distance (and hence location) of the medium discontinuity that caused the reflection. Meanwhile, the nature of the reflection (waveform) can be used to reveal information about the nature of the discontinuity. Practical devices are built around these concepts in many application fields. Among those devices are RADARS that are used to locate, monitor, and identify airplanes and other flying objects. Sonars are other devices that have applications in medical applications as well as beneath the earth explorations. The most direct application in transmission lines is the Time Domain Refelctometer, TDR, used for cable testing and fault location. The optical version of that is called OTDR (Optical Time Domain Refelctometer) which is used for fiber cables.

With this background in mind, we can see the importance of discussing time domain bounce-diagrams even though we will limit the analysis to lossless media. Furthermore, we will initially confine our discussion to pure resistive discontinuities including the source and load “impedances”.

Time Domain Bounce Diagram for Lossless Lines and Resistive Discontinuities:

In this discussion we will be working with the signals in their time domain form, and hence, we will be using the lower case notations, $v(z, t)$ and $i(z, t)$. Furthermore, we will be dealing with time domain (local/instantaneous) reflection coefficients for which we will use the notation ρ instead of Γ. This local reflection coefficient can only be found at the discontinuities at the instant the “incident” signal reaches that discontinuity. This concept is demonstrated in Figure 2.31. At any instance of time, and in locations with no discontinuities, only one signal may exist. However, at discontinuity locations, reflections off the discontinuity result in two coexisting signals, the incident signal, and its reflection.

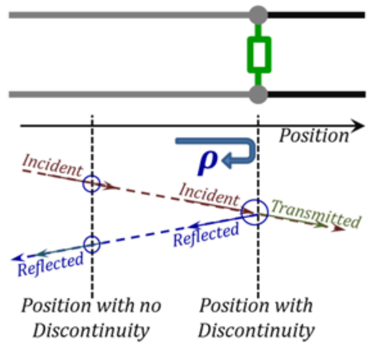


Figure 2.31

Now, consider the setup shown in Figure 2.32 with a source and load as shown. Both source and load impedances (R_S *and* R_L) are purely resistive. The transmission line connecting the two sides is assumed lossless, and hence has a pure resistive characteristic impedance R_o. Consequently, we can write the local reflection coefficients at the source and load ends of the line as:

$$\rho_S = \frac{[R_S - R_o]}{[R_S + R_o]} \quad and \quad \rho_L = \frac{[R_L - R_o]}{[R_L + R_o]} \qquad (2.96)$$

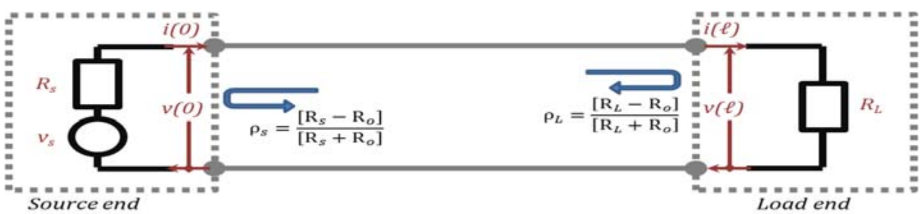


Figure 2.32

Also, the initial launch $a(t) = v(z = 0, t = 0)$ can be written as the voltage divider of v_s(t) between the line's characteristic resistance (impedance), R_o, and the source's resistance, R_s, see Figure 2.33:

$$a(t) = v(0)|_{t=0} = v_s \frac{R_o}{[R_s + R_o]} \qquad (2.97)$$

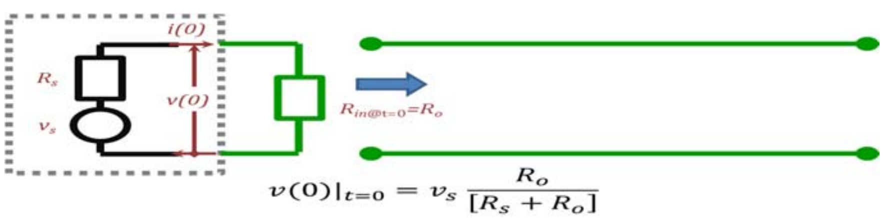


Figure 2.33

Now, we can construct the time-domain bounce diagram as we follow the signal travel (delay) and reflect (and transmit) at discontinuities (load and source resistances). As discussed earlier, Equation (2.40), the time delay corresponding to a distance travelled is given by $\tau_{distance}$=distance/c_{ph}.

Using the notation, $\tau = \ell/c_{ph}$, $\tau_z = z/c_{ph}$, and $\tau_d = (\ell\text{-}z)/c_{ph}$, the first positive z waveform is the initial launch a(t) which becomes a(t- τ_z) after traveling a distance z and a(t- τ) at the load. The first reflection at the load yields [ρ_L].[a(t- τ)] as the first negative z traveling wave. This reflection travels back towards the source experiencing further delays to arrive at the source after a delay of $2\ell/c_{ph} = 2\tau$, and hence can be written as $[\rho_L].[a(t-2\tau)]$. Reflecting from the source, a second positive z travel is initiated through a reflection coefficient of $[\rho_s]$. This signal is thus initiated as $[\rho_s].[\rho_L].[a(t-2\tau)]$ and travels towards the load accumulating more delays. The multiple reflections go back and forth with further delays (time shift) of distance/c_{ph} and amplitudes changing by the reflection coefficients multipliers $[\rho_s]$ and $[\rho_L]$.

Now, we examine the transmission (through) waveforms that take place at the load and the source ends upon each reflection. Upon the arrival of the signal at the discontinuity, a reflection voltage develops at that discontinuity in the form of $[\rho_{s\text{ or }L}]$.[incident voltage]. The total voltage at that location is then equal to [incident voltage]+$[\rho_{s\text{ or }L}]$.[incident voltage]=$[1+\rho_{s\text{ or }L}]$.[incident voltage]. This voltage transmits to the other side of the discontinuity whether it is the source or the load. Consequently, the first arrival signal to the load side is of the form $[a(t-\tau)]+[\rho_L].[a(t-\tau)]= [1+\rho_L].[a(t-\tau)]$ and the first arrival to the source is $[\rho_L].[a(t-2\tau)]+[\rho_s].[\,[\rho_L].[a(t-2\tau)]\,] =[1+\rho_s].[\rho_L].[a(t-2\tau)]$.

The process we just discussed in the above paragraphs is demonstrated in Figure 2.34 and the resulting waveforms are summarized in Table 2.4.

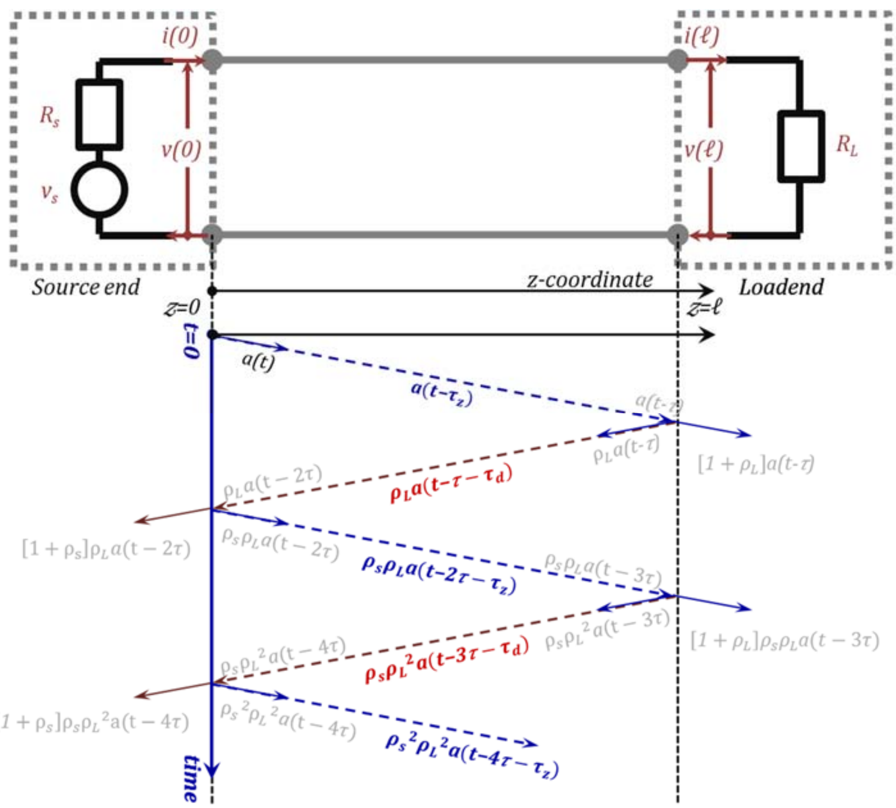


Figure 2.34

Table 2.4

@ Source End			@ location z, ($\mathbf{d} = \boldsymbol{\ell} - \mathbf{z}$)		@ Load End		
		+z		+z		+z	
	-z		-z		-z		To $\boldsymbol{Z_L}$
To $\boldsymbol{Z_s}$							

	$a(t)$	$a(\boldsymbol{t}-\boldsymbol{\tau_z})$	$a(t-\tau)$
$\rho_L a(t-2\tau)$	$\boldsymbol{\rho_L a(t-\tau-\tau_d)}$	$\rho_L a(t\text{-}\tau)$	$[1+\rho_L]a(t-\tau)$
$[1+\rho_s]\rho_L a(t-2\tau)$	$\rho_s\rho_L a(t-2\tau)$	$\boldsymbol{\rho_s\rho_L} a(\boldsymbol{t}\text{-}2\tau-\boldsymbol{\tau_z})$	$\rho_s\rho_L a(t-3\tau)$
$\rho_s{\rho_L}^2 a(t-4\tau)$	$\boldsymbol{\rho_s{\rho_L}^2 a(t-3\tau-\tau_d)}$	$\rho_s{\rho_L}^2 a(t-3\tau)$	$[1+\rho_L]\rho_s\rho_L a(t-3\tau)$
$[1+\rho_s]\rho_s{\rho_L}^2 a(t-4\tau)$	${\rho_s}^2{\rho_L}^2 a(t-4\tau)$	$\boldsymbol{{\rho_s}^2{\rho_L}^2 a(t-4\tau-\tau_z)}$	${\rho_s}^2{\rho_L}^2 a(t-5\tau)$
${\rho_s}^2{\rho_L}^3 a(t-6\tau)$	$\boldsymbol{{\rho_s}^2{\rho_L}^3 a(t-5\tau-\tau_d)}$	${\rho_s}^2{\rho_L}^3 a(t-5\tau)$	$[1+\rho_L]{\rho_s}^2{\rho_L}^2 a(t-5\tau)$
$[1+\rho_s]{\rho_s}^2{\rho_L}^3 a(t-6\tau)$	${\rho_s}^3{\rho_L}^3 a(t-6\tau)$		

		Time Domain Series
@Source	*To* $\boldsymbol{Z_s}$	$a(t)+[1+\rho_s]\rho_L\cdot\{a(t-2\tau)+\rho_s\rho_L a(t-4\tau)+{\rho_s}^2{\rho_L}^2 a(t-6\tau)+\cdots\}$
	-z	$\rho_L\cdot\{a(t-2\tau)+\rho_s\rho_L a(t-4\tau)+{\rho_s}^2{\rho_L}^2 a(t-6\tau)+\cdots\}$
	+z	$a(t)+\rho_s\rho_L a(t-2\tau)+{\rho_s}^2{\rho_L}^2 a(t-4\tau)+{\rho_s}^3{\rho_L}^3 a(t-6\tau)+\cdots$
@z	*-z*	$\boldsymbol{v^-(z,t)=\rho_L\cdot\{a(t-\tau-\tau_d)+\rho_s\rho_L a(t-3\tau-\tau_d)+{\rho_s}^2{\rho_L}^2 a(t-5\tau-\tau_d)+\cdots\}}$
	+z	$\boldsymbol{v^+(z,t)=a(t-\tau_z)+\rho_s\rho_L a(t-2\tau-\tau_z)+{\rho_s}^2{\rho_L}^2 a(t-4\tau-\tau_z)}$ $\boldsymbol{+{\rho_s}^3{\rho_L}^3 a(t-6\tau-\tau_z)+\cdots}$
@Load	*-z*	$\rho_L\cdot\{a(t-\tau)+\rho_s\rho_L a(t-3\tau)+{\rho_s}^2{\rho_L}^2 a(t-5\tau)+\cdots\}$
	+z	$a(t-\tau)+\rho_s\rho_L a(t-3\tau)+{\rho_s}^2{\rho_L}^2 a(t-5\tau)+\cdots$
	To $\boldsymbol{Z_L}$	$[1+\rho_L]\cdot\{a(t-\tau)+\rho_s\rho_L a(t-3\tau)+{\rho_s}^2{\rho_L}^2 a(t-5\tau)+\cdots\}$

From table 2.4, we can write expressions for the signal as a function of position and time as:

$$v^+(z,t)=a(t-\tau_z)+\rho_s\rho_L a(t-2\tau-\tau_z)+{\rho_s}^2{\rho_L}^2 a(t-4\tau-\tau_z)+{\rho_s}^3{\rho_L}^3 a(t-6\tau-\tau_z)+\cdots \qquad (2.98)$$

$$v^-(z,t)=\rho_L\cdot\{a(t-\tau-\tau_d)+\rho_s\rho_L a(t-3\tau-\tau_d)+{\rho_s}^2{\rho_L}^2 a(t-5\tau-\tau_d)+\cdots\} \qquad (2.99)$$

$$v(z,t)=v^+(z,t)+v^+(z,t) \qquad (2.100)$$

Time Domain Reflectometry and the Bounce Diagram:

As discussed earlier, time-domain reflectometry is a technique used for extracting information of a transmission medium by monitoring and analyzing reflections off the medium due to an incident pulse. Typical pulse shapes used in TDR devices are either a step-like waveform (a practical version of an ideal step function, u(t), or an impulse (a practical version of a Dirac-delta function, δ(t)). In this section, we will work with the ideal waveforms for mathematical convenience.

Time-Domain Reflectometry for Ideal Step Waveform Excitations:

In this case, we assume $v_s(t)$=V u(t), where V is the amplitude of the voltage step. To demonstrate the concept, let us assume the simple network of Figure 2.34 with the following parameters:

Vs = 1 u(t) Volts

Ro = 75 Ohms

RL = 125 Ohms

RS = 50 Ohms

Line length = 15 cm

Speed of light in TL material = 1.5 E8 m/s

Using these parameters, we conclude that a(t) = 1 u(t) (75)/(75+50) = 0.6 u(t) Volts

ρ_s = (50-75)/(50+75) = -0.2

ρ_L = (125-75)/(125+75) = 0.25

τ = 0.15/1.5E8 = 1 ns

Next, we use these values to substitute in the bounce diagram of Figure 2.34 and extract the essential information from Table 2.4, namely, the TDR (time-domain reflectometry) signal (appearing at the source end, and TDT (time domain transmission) signal appearing at the load end. The results are shown below:

	Time Domain Series
@Source end (TDR)	$0.6u(t) + 0.12u(t-2) - 0.006u(t-4) + 0.0003u(t-6)$
@Load end (TDT)	$0.75u(t-1) - 0.0375u(t-3) + 0.001875u(t-5) + \cdots$

Consequently, the TDR and TDT waveforms are as shown in Figure 2.35 below:

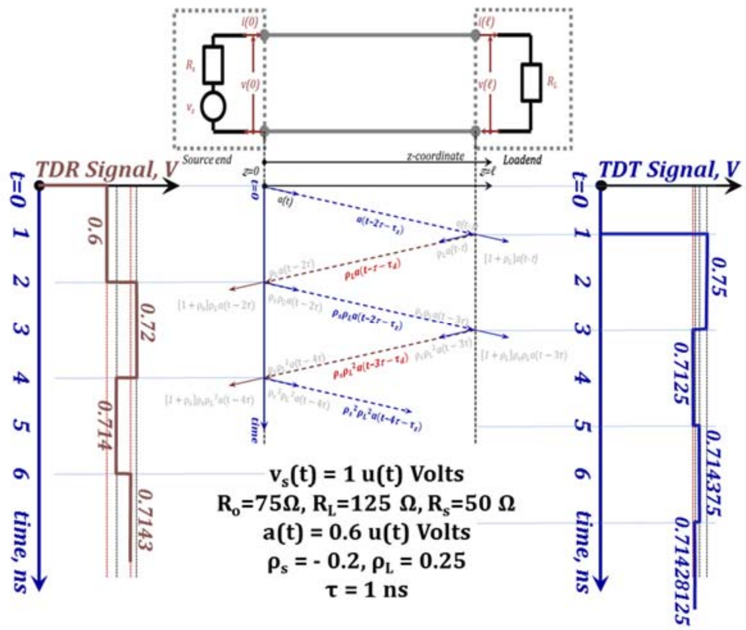


Figure 2.35

It is worth noting that both TDR and TDT signals converge to the same amplitude. As time approaches infinity, the "final value theorem" tells us that the solution is the same as the DC solution. For DC, the TL acts like a zero length wire, and hence both voltages at the sending and receiving ends are the same as the voltage divider between the load and source impedances:

$V_{t=\infty} = V\, R_L/(R_s+R_L) = (1\,).(125)/(50+125) = 0.7142857\ V$

Time-Domain Reflectometry for Ideal Dirac-Delta Impulse Waveform Excitations:

To demonstrate, let us consider the same example we just did for the step waveform while considering Vs = 1 δ(t) volts, hence a(t) = 0.6 δ(t) Volts and the expressions for the TDR and TDT signals becomes:

	Time Domain Series

@Source end (TDR)	$0.6\delta(t) + 0.12\delta(t-2) - 0.006\delta(t-4) + 0.0003\delta(t-6)$
@Load end (TDT)	$0.75\delta(t-1) - 0.0375\delta(t-3) + 0.001875\delta(t-5) + \cdots$

Consequently, the TDR and TDT waveforms are as shown in the Figure 2.36 below:

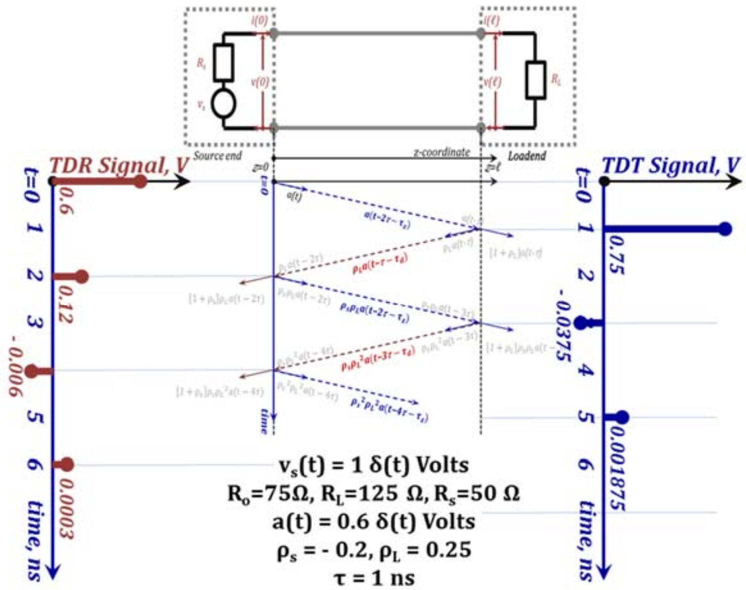


Figure 2.36

The final value in this case is a zero since δ(t) vanishes when t approaches infinity.

Addendum E

The Smith Chart

The Smith Chart:

It was evident from the presentation of the previous sections that the impedance analysis is of prime importance in studying TL performance. However, the corresponding equations looked a bit complicated and are not easy to visualize. TL and RF circuit designers need a tool to enable visual assessment and prediction of problem areas and assessing impedance levels to work on impedance matching problems; such a tool exists and is known as the Smith Chart.

The Smith chart is a graphical tool that maps the impedance values vs. the corresponding reflection coefficient values. Since both are complex numbers, we don't expect a single trace of one vs the other, instead, the Smith chart is composed of multiple traces with parametric variables.

The Smith chart is simply the graph that describes the mathematical relationship between the normalized impedance $Z(z\ or\ d)/Z_o\ and\ \Gamma(z\ or\ d)$. Using Equation (2.46), we can write:

$$Z_n(z\ or\ d) = \frac{Z(z\ or\ d)}{Z_o} = \frac{[1+\Gamma(z\ or\ d)]}{[1-\Gamma(z\ or\ d)]} \quad and \quad \Gamma(z\ or\ d) = \frac{[Z_n(z\ or\ d)-1]}{[Z_n(z\ or\ d)+1]} \qquad (2.101)$$

Spelling out all complex quantities in terms of their real and imaginary components

$$Z_n(z\ or\ d) = \frac{R(z\ or\ d)+jX(z\ or\ d)}{Z_o} = r(z\ or\ d) + jx(z\ or\ d)\ ,\ and$$

$$\Gamma(z\ or\ d) = \Gamma_{re}(z\ or\ d) + j\Gamma_{im}(z\ or\ d)$$

By dropping the $z\ or\ d$ notation for convenience,

$$r + jx = \frac{[1 + \Gamma_{re} + j\Gamma_{im}]}{[1 - \Gamma_{re} - j\Gamma_{im}]}$$

Equating both the real and imaginary parts of the above equation yields

$$r = \frac{[1 - (\Gamma_{re})^2 - (\Gamma_{im})^2]}{[(1 - \Gamma_{re})^2 + (\Gamma_{im})^2]}$$

$$x = \frac{[2\Gamma_{im}]}{[(1 - \Gamma_{re})^2 + (\Gamma_{im})^2]}$$

Manipulation of these two equations yields the following convenient forms:

$$\left[\Gamma_{re} - \frac{r}{1+r}\right]^2 + [\Gamma_{im}]^2 = \left[\frac{1}{1+r}\right]^2 \qquad (2.102)$$

$$[\Gamma_{re} - 1]^2 + \left[\Gamma_{im} - \frac{1}{x}\right]^2 = \left[\frac{1}{x}\right]^2 \qquad (2.103)$$

Both equations describe families of circles on a plane formed by $\Gamma_{re}\ and\ \Gamma_{im}$ as its horizontal and vertical axes.

The general circle equation centered at $(\Gamma_{re,o}, \Gamma_{im,o})\ with\ a\ radius\ \Gamma_{radius}$ is this plane takes the form:

$$\left[\Gamma_{re} - \Gamma_{re,o}\right]^2 + \left[\Gamma_{im} - \Gamma_{im,o}\right]^2 = [\Gamma_{radius}]^2$$

Hence, for the first Equation (2.102), the "*r*" circles will have their origins located at $(\frac{r}{1+r}, 0)$ and the corresponding radius is given by $(\frac{1}{1+r})$, while for the "*x*" circles, Equation (2.103), the origins are located at $(1, \frac{1}{x})$ and the corresponding radius is given by $(\frac{1}{x})$., see Figure 2.37. In Figure 2.38 shows sets of these circles for various values of r and x.

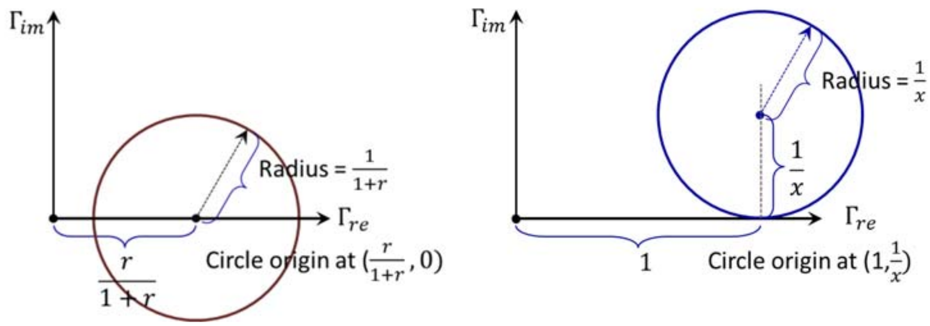


Figure 2.37

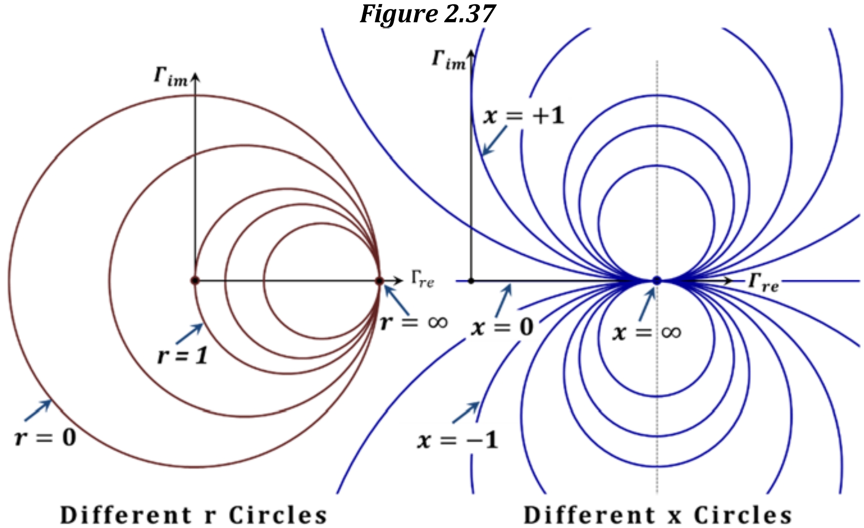


Figure 2.38

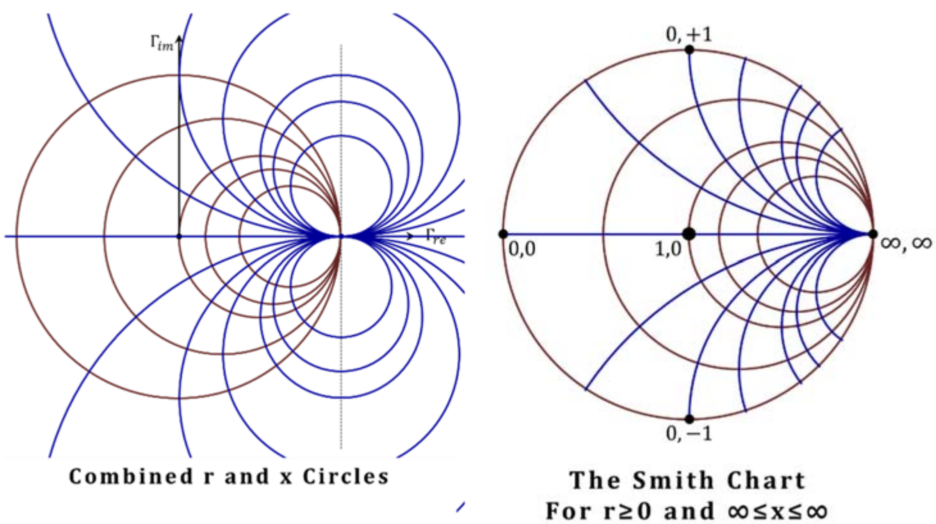


Figure 2.39

In the left part of Figure 2.39, both the r and x circles are assembled on the same chart. Finally, the chart on the right is the familiar Smith chart that appears in most prints. In that chart, the parts corresponding to r<0 are eliminated.

Scales on the Smith Chart:

There are several scales typically included in a printed Smith chart that can be used to facilitate data entry and extraction. In this section, we will discuss the main scales, which are displayed in Figure 2.40.

1. *The magnitude of the reflection coefficient (|Γ|) scale:* The magnitude of *Γ* is the length of the phasor originating from the origin of the chart and hence is the radius of the circle centered at the origin and passes by the point representing the phasor. The scale for *|Γ|* is typically positioned on the bottom left of the chart with its zero aligned vertically with the chart origin and has the same "unit length" as that of the "r=0" circle. (Note that r=0 implies *|Γ|=1).*

To demonstrate the use of this scale, an example of |Γ|=0.76 is shown in the figure. From the location of |Γ|=0.76 on the scale a vertical line is drawn till it intercepts the horizontal axis of the chart. Next, a circle (centered at the origin) is drawn to pass by the point of interception. Consequently, the radius of this circle is 0.76, and all points on this circle will have |Γ|=0.76.

2. *The phase angle of the reflection coefficient (∠Γ) scale:* The phase angle of *Γ* is the angle the *Γ* phasor makes with the "horizontal" Γ_{re} axis. The scale for *∠Γ* is positioned on the outer perimeter of a circle outside the "r=0" circle of the chart. The scale is calibrated from -180° to +180° with its zero aligned with the positive side of the horizontal axis of the chart.

To demonstrate the use of this scale, Assume a phase angle of +72° for the |Γ|=0.76 example discussed earlier. As shown in the figure, a radial line is placed on the chart for the origin to the +72° mark on the scale. The point of intercept represents the Γ phasor 0.76*∠+72°*.

3. *Normalized distance moved scale:* This scale has to do with the relative locations of the Γ phasors corresponding to different locations on the line. Referring to Figure 2.40, $\Gamma(z_2)$ and $\Gamma(z_1)$ can be related to each other through the use of Equation (2.43):

$$\Gamma(z_1) = \frac{V^- e^{+\gamma z_1}}{V^+ e^{-\gamma z_1}} = \Gamma(0) \cdot e^{+2\gamma z_1} \qquad (2.104)$$

$$\Gamma(z_2) = \Gamma(0) \cdot e^{+2\gamma z_2} = \Gamma(z_1) \cdot e^{+2\gamma (z_2 - z_1)} \qquad (2.105)$$

$$Hence, |\Gamma(z_2)| = |\Gamma(z_1)| e^{+2\alpha\,(z_2 - z_1)} \; and \; \angle\Gamma(z_2) = \angle\Gamma(z_1) + 2\beta(z_2 - z_1),\ or$$

$$|\Gamma(z_2)| = |\Gamma(z_1)| e^{+2\alpha\,(z_2 - z_1)} \; and \; \angle\Gamma(z_2) = \angle\Gamma(z_1) + 4\pi \cdot \left[\frac{(z_2 - z_1)}{\lambda}\right] \qquad (2.106)$$

Consequently, moving on the line from z_1 to z_2 implies a change of the angle of the phasor (rotation) by $4\pi \cdot [(z_2 - z_1)/\lambda]$. If $z_2>z_1$, the change is an increase in the angle and hence a counter-clockwise rotation, and for $z_2<z_1$, the rotation is clockwise. One can use the phase angle scale to enter these rotations on the chart, however, two normalized distance scales are given to facilitate this process without multiplying the 4π by the $[(z_2 - z_1)/\lambda]$. The normalized distance scales provide direct calibration of the $[(z_2 - z_1)/\lambda]$ quantity. One of these scales increases counter-clockwise to enable tracking "increasing z" locations as we move "towards" the load, while the other scale increases clockwise for tracking "decreasing z" locations (increasing d) "towards" the source (generator). Both scales are provided along the perimeter of a circle located outside the phase angle scale with the "Towards Generator" scale marked on the

outer side of the circle and the "Towards Load" scale marked on the inner side of the circle.

Both scales carry tick marks from 0 to 0.5 in a full circle, 360°. This is because a $\lambda/2$ distance change correspond to $4\pi \cdot [(z_2 - z_1)/\lambda] = 2\pi$ which is 360°.

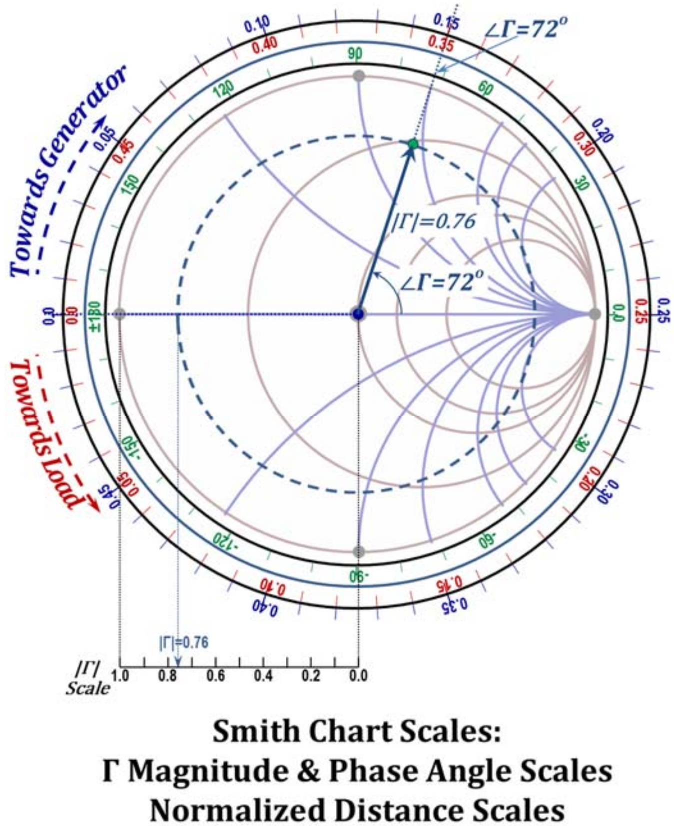


Figure 2.40

Transmission Line Trace on the Smith Chart:

The Smith chart is a graphical representation that can enable us to monitor the transmission line performance parameters in terms of its Γ(z) and Z(z) at different location of the line. Referring to Figure 2.41, we will demonstrate how the chart can be used to achieve this objective.

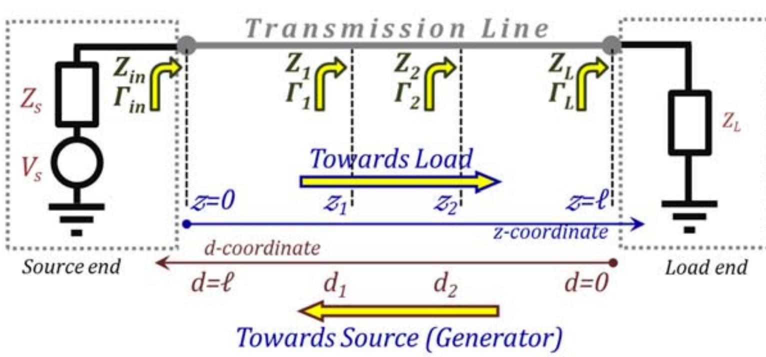


Figure 2.41

Since the chart is a plot of the Γ phasor, then tracking the line trace implies tracing its Γ(z) on the chart. Using Equation (2.105), we can write the expression for the Γ(z) in terms of Γ(0) and Γ(ℓ) as:

$$\Gamma(z) = \Gamma(0) \cdot e^{+2\gamma z} = \Gamma(0) \cdot e^{+2\alpha z} e^{+2j\beta z} = \Gamma(0) \cdot e^{+2\alpha z} e^{+j4\pi\cdot(z/\lambda)} \qquad (2.107)$$

and,

$$\Gamma(z) = \Gamma_L \cdot e^{-2\gamma(\ell - z)} = \Gamma_L \cdot e^{-2\alpha(\ell-z)} e^{-j4\pi\cdot[(\ell-z)/\lambda]}$$

The latter can be rewritten as a function of d, (the distance from the load end), using d=ℓ-z:

$$\Gamma(d) = \Gamma_L \cdot e^{-2\gamma d} = \Gamma_L \cdot e^{-2\alpha d} e^{-2j\beta d} = \Gamma_L \cdot e^{-2\alpha d} e^{-j4\pi\cdot(d/\lambda)} \qquad (2.108)$$

Equations (2.107) and (2.108) can be rewritten in terms of their magnitude and phase components:

$$|\Gamma(z)| = |\Gamma(0)| e^{+2\alpha z} \ \ and \ \angle\Gamma(z) = \angle\Gamma(0) + 4\pi \cdot (z/\lambda) \qquad (2.109)$$

and

$$|\Gamma(d)| = |\Gamma_L| e^{-2\alpha d} \ \ and \ \angle\Gamma(d) = \angle\Gamma_L - 4\pi \cdot (d/\lambda) \qquad (2.110)$$

Equations (2.109) and (2.110) are telling us that the magnitude of the Γ phasor changes with position on the line depending on the attenuation coefficient, α. They also show us the continuous (linear) phase change of the phasor.

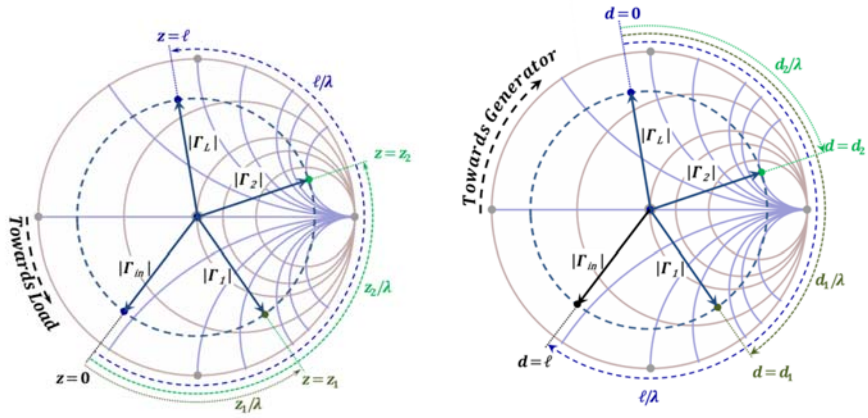
Case of Lossless TL, α=0:

In this case, and as can be shown from Equations (2.109) and (2.110), the magnitude of the reflection coefficient is the same (constant) at all line locations:

$$|\Gamma(z)| = |\Gamma(0)| = |\Gamma_L|$$

and, hence, the trace of |Γ(z)| is a circle centered at the origin and has a radius of |Γ(0)|= |Γ_L|. On the other hand, the phase angle increases as we move towards the load and decreases with motion towards the source. Consequently, the left chart of Figure 2.42 displays the trace of Γ(z) starting at z=0, moving towards the load, passing by points z_1 and z_2 till the end of the line at z=ℓ. The phase shifts from z=0 to z=z_1, z=z_2, and finally z=ℓ are represented by angles of counter-clockwise rotations starting at the phase at z=0 in the amounts 4π (z_1/λ), 4π (z_2/λ), 4π (ℓ/λ), respectively. These rotation angles correspond to rotations of (z_1/λ), (z_2/λ), and (ℓ/λ), respectively, on the counter-clockwise normalized distance scale labeled “Towards Load”.

Likewise, the right chart in Figure 2.42 demonstrates the same line but starting the trace at the load end on the TL. The trace is thus starting at d=0 (z=ℓ), and moves towards the source, passing by points d_2 (or z_2) and d_1 (or z_1) till the source end of the line at d=ℓ (z=0). The phase shifts from d=0 to d=d_2, d=d_1, and finally d=ℓ are represented by angles of clockwise rotations in the amounts 4π (d_2/λ), 4π (d_1/λ), 4π (ℓ/λ), respectively. Again, these rotation angles correspond to rotations of (d_2/λ), (d_1/λ), and (ℓ/λ), respectively, on the clockwise normalized distance scale labeled “Towards Generator”.



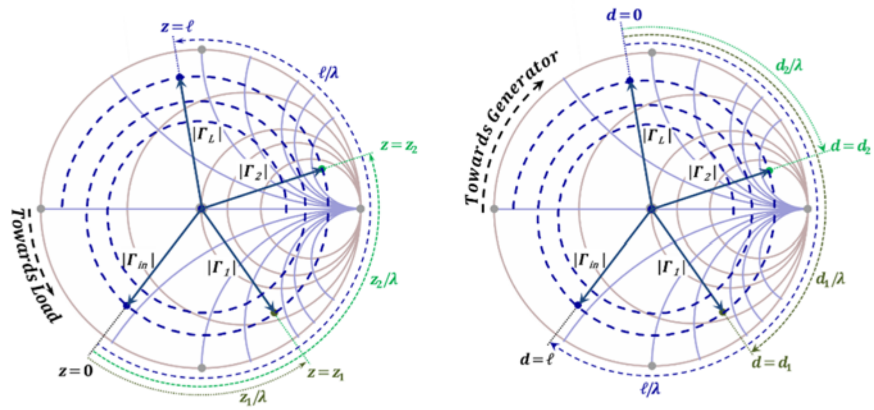
Trace of a Lossless TL
Constant |Γ| Circle

Trace of a Lossless TL
Constant |Γ| Circle

Figure 2.42

Case of Lossy TL, α≠0:

In this case, and as can be shown from Equations (2.109) and (2.110), the magnitude of the reflection coefficient varies with position, and hence, the Γ(z) is no longer a circle of a constant radius. Varying the magnitude while rotating the phase results in a spiral trace as demonstrated in Figure 2.43. The prescribed spiral pitch depends on the value of the attenuation coefficient. For a low loss line, the pitch is small while it becomes large for larger attenuation coefficients. The spiral trace is inwards as we move towards the source while appearing outwards for motions towards the load.



Trace of a Lossy TL
|Γ| Spirals out to Load

Trace of a Lossy TL
|Γ| Spirals in to Source

Figure 2.43

Figure 2.43 demonstrates the same TL traces discussed earlier in Figures 2.41 and 2.42 in the presence of line loss, α≠0.

How Does the Smith Chart Work?

Starting with Z(z or d):

In this case, we assume that the driving point impedance is known at some point on the line located at a distance z from its source end (distance d from its load end). The objective is to determine the frequency-domain reflection coefficient at that location. The mathematical expression to be used in this case if we were to do the calculations mathematically is that of Equation (2.46 or 101):

$$\Gamma(z\ or\ d) = \frac{[Z(z\ or\ d) - Z_o]}{[Z(z\ or\ d) + Z_o]}$$

The Smith chart (graphical) procedure goes as follows, refer to Figure 2.44:

1. Choose a normalization impedance, Z_o, for the problem at hand. Typically, the characteristic impedance of the "dominant" transmission line of the circuit is a preferred choice.
2. Compute the normalized impedance by dividing $\frac{Z(z\ or\ d)}{Z_o} = Z_n(z\ or\ d) = r + jx$
3. Locate the "*r*" circle for the obtained "r" value
4. Locate the "*x*" circle for the obtained "x" value
5. The intersection of the two circles defines the $\Gamma(z\ or\ d)$ point
6. The line connecting the origin to the $\Gamma(z\ or\ d)$ point is the phasor representing $\Gamma(z\ or\ d)$
7. The length of the $\Gamma(z\ or\ d)$ phasor is the magnitude of $\Gamma(z\ or\ d)$.
8. Draw a circle centered at the origin passing by the Γ point and project the radius of this circle on the $|\Gamma|$ scale to read the magnitude of Γ.
9. The angle the $\Gamma(z\ or\ d)$ phasor makes with the horizontal (real axis) is the angle of the $\Gamma(z\ or\ d)$ phasor. Read the phase angle of Γ on the provided angle scale.
10. In rectangular format, the projections of the $\Gamma(z\ or\ d)$ phasor on the horizontal and vertical axes are the real and imaginary parts of the $\Gamma(z\ or\ d)$ phasor, respectively.

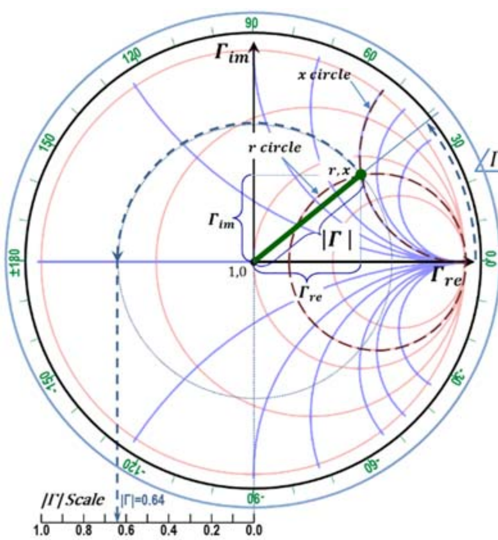


Figure 2.44

Starting with the $\Gamma(z\ or\ d)$ phasor:

In this case, we assume that the frequency-domain reflection coefficient is known at some point on the line located at a distance z from its source end (distance d from its load end). The objective is to determine the driving point impedance of the line at that location. The mathematical expression to be used in this case if we were to do the calculations mathematically is that of Equation (2.46 or 101):

$$Z(z\ or\ d) = Z_o \frac{[1 + \Gamma(z\ or\ d)]}{[1 - \Gamma(z\ or\ d)]}$$

The Smith chart (graphical) procedure goes as follows, again, refer to Figure 2.44:

1. The normalization impedance Z_o must be given to us along with the Γ(z or d).
2. Locate the $\Gamma(z\ or\ d)$ phasor (Magnitude and Phase) or Real and Imaginary.
3. For |Γ|, Γ_{re} and Γ_{im} use the |Γ| Scale provided to enter the proper line lengths to be graphed on the chart. For phase, use the phase angle scale to enter the $\underline{/\Gamma}$.
4. Locate the "r" circle that passes by the $\Gamma(z\ or\ d)$ phasor point – read the value of r.
5. Locate the "x" circle that passes by the $\Gamma(z\ or\ d)$ phasor point – read the value of x.
6. Compose the obtained r and x values to obtain a numerical value for the complex normalized impedance $Z_n(z\ or\ d) = Z/Z_o = r + jx$.
7. De-normalize $Z_n(z\ or\ d)$ to compute the actual driving point impedance of the line $Z(z\ or\ d)$ at the position (z or d). To de-normalize, you multiply the normalized impedance by the given normalization impedance Z_o. Thus, $Z(z\ or\ d) = Z_n(z\ or\ d) \cdot Z_o$.

Finding $\Gamma(z_2\ or\ d_2)$ (and $Z(z_2\ or\ d_2)$) knowing $\Gamma(z_1\ or\ d_1)$ or $Z(z_1\ or\ d_1)$

In this case, we assume that line information (either Z or Γ) is known at some point on the line (e.g. z_1 or d_1). The objective is to determine the corresponding Z and Γ at another point on the line (e.g. z_2 or d_2). The mathematical expressions for this case was given earlier in Equation (2.106), i.e.:

$$|\Gamma(z_2)| = |\Gamma(z_1)|e^{+2\alpha\,(z_2 - z_1)}\ and\ \angle\Gamma(z_2) = \angle\Gamma(z_1) + 4\pi \cdot [(z_2 - z_1)/\lambda] \qquad (2.106)$$

Once $\Gamma(z_2)$ is determined, we would use Equation (2.46) to get $Z(z_2)$.

The Smith chart (graphical) procedure is illustrated in Figures 2.45:

1. Locate the line position z_1 (or d_1) on the chart though the Γ Phasor or impedance Z information as demonstrated earlier.
   - Moving toward the load, $z_2 > z_1$, then according to Equation (2.106), $|\Gamma(z_2)| > |\Gamma(z_1)|$ and $and\ \angle\Gamma(z_2) = \angle\Gamma(z_1)$
     - The corresponding trace for line points on the chart is a counter-clockwise growing spiral. The spiral radius growth is determined by the exponent $e^{+2\alpha\,(z_2 - z_1)}$. This computation is to be performed manually since the spiral is not part of the chart.
   - However, if $z_2 < z_1$ $(or\ d_2 > d_1)$, i.e. we move towards the source (generator), the corresponding motion on the (same) spiral would be in the opposite direction.
2. Compute the normalized distance of rotation $[(z_2 - z_1)/\lambda]$ (or $[(d_2 - d_1)/\lambda]$) and enter this amount on the chart using the normalized distance scale starting at point "z_1" to locate point "z_2".
3. Read $\Gamma(z_2)$ (and $Z_{n2} = r_2 + jx_2$)

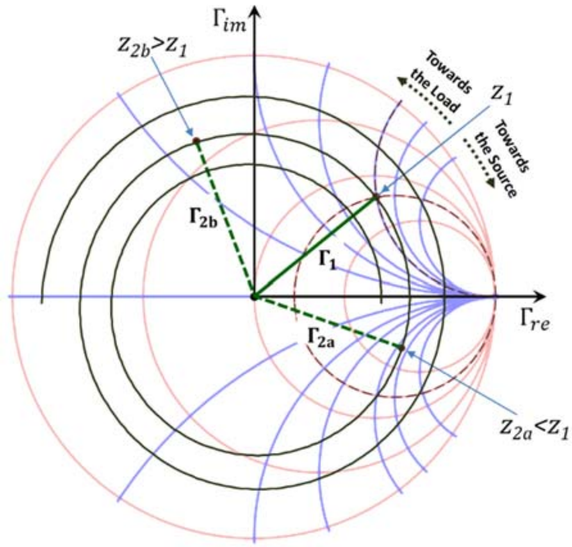


Figure 2.45

<u>*Special case (Lossless TL),* $\alpha = 0$:</u>

This case is similar to the case above except for having a constant radius circle replacing the spiral trace. Figure 2.46 demonstrates the graphical procedure on the chart in a similar manner.

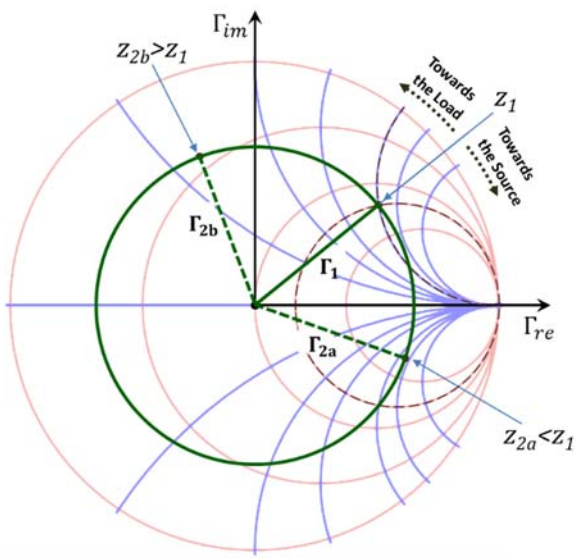


Figure 2.46

The Admittance Smith chart:

At high frequencies, it is often preferred to work with admittances instead of impedances. One reason is that most circuit components are mounted in parallel with one terminal grounded. Series components are floating devices and would lead to the circuit suffering from additional undesired parasitic. Working with shunt devices makes admittance expressions more attractive since combining admittances in parallel is achieved through simple addition of their values.

Examining expression (2.101), we can rewrite it in terms of admittance components as follows:

$$Z_n(z\ or\ d) = \frac{Z(z\ or\ d)}{Z_o} = \frac{[1+\Gamma(z\ or\ d)]}{[1-\Gamma(z\ or\ d)]} \quad and \quad \Gamma(z\ or\ d) = \frac{[Z_n(z\ or\ d)-1]}{[Z_n(z\ or\ d)+1]} \qquad (2.101)$$

$$Y_n(z\ or\ d) = \frac{Y(z\ or\ d)}{Y_o} = \frac{1}{Z_n(z\ or\ d)} = \frac{[1-\Gamma(z\ or\ d)]}{[1+\Gamma(z\ or\ d)]} \quad and \quad -\Gamma(z\ or\ d) = \frac{[Y_n(z\ or\ d)-1]}{[Y_n(z\ or\ d)+1]} \qquad (2.111)$$

Comparing Equations (2.101) and (2.111), we notice the similarity except for replacing the Z_n by Y_n and at the same time replacing Γ by -Γ. Likewise, if we express Y_n in terms of its real and imaginary conductance g_n and susceptance b_n, $Y_n= g_n+jb_n$, we can show that the constant g and b circles are of the form:

$$\left[-\Gamma_{re} - \frac{g}{1+g}\right]^2 + [-\Gamma_{im}]^2 = \left[\frac{1}{1+g}\right]^2 \qquad (2.112)$$

$$[-\Gamma_{re} - 1]^2 + \left[-\Gamma_{im} - \frac{1}{b}\right]^2 = \left[\frac{1}{b}\right]^2 \qquad (2.113)$$

which are, again, similar to (2.102) and (2.103) by replacing r and x by g and b and at the same time replacing Γ_{re} and Γ_{im} by $-\Gamma_{re}$ and $-\Gamma_{im}$, respectively.

The conclusion is that for a given Γ phasor on the chart (corresponding to a given value of Z_n=r+jx), the –Γ phasor on the same chart corresponds to the Y_n=g+jb value at that location on the line.

This concept is demonstrated in the chart of Figure 2.47. The Γ phasor is shown for the normalized impedance Z_n=01.5+j2.0. Thus, the corresponding –Γ phasor reveals to us the corresponding Y_n which is Y_n=0.24-j0.32. (It is interesting to observe that the chart can be used as a complex number inverter since $Y_n=1/Z_n$. Hence, if you place a complex number as the Z_n, the corresponding Y_n point is the inverse of that complex number.)

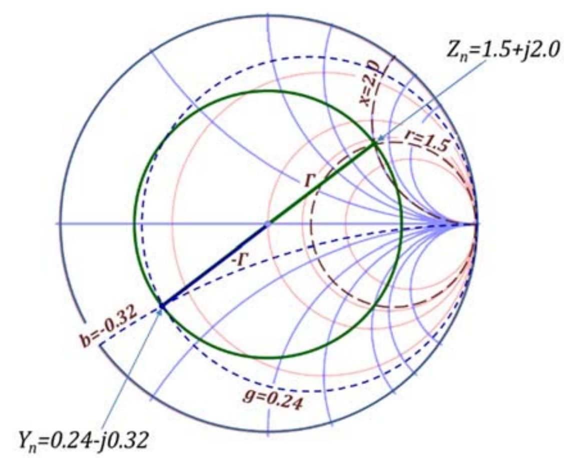


Figure 2.47

Now that we identified the placement of admittances on the chart, we can actually work out series and parallel combinations in conjunction with segments of transmission lines in a relatively convenient way. To demonstrate, let us consider the example of Figure 2.48.

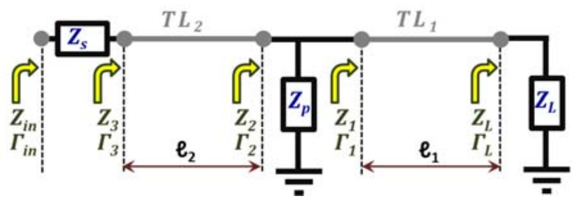


Figure 2.48

The figure shows a two-transmission line segments TL_1 and TL_2, both having the same Z_o and γ but different lengths. The two lines are connected in cascade with a shunt impedance Z_p in between. TL_1 is terminated in Z_L while a series impedance Z_s is connected at the input side of TL_2. It is desired to find the combined circuit input impedance Z_{in} and reflection coefficient phasor Γ_{in}.

To solve this problem analytically, we would start with TL_1 terminated in Z_L. Using Equation (2.71), we obtain the input impedance to that segment, Z_1. Next, we add Z_p in parallel to Z_1 to get Z_2. Again, we use Equation (2.71) for TL_2 terminated in Z_2 as a load and we get its input impedance Z_3, and finally, add Z_s in series to get Z_{in}. To get Γ_{in}, we use Equation (2.46).

To do this problem using the Smith Chart, we follow the procedure outlined below and demonstrated in Figure 2.49.

1. Choose the normalization impedance to be Z_o of the TLs.
2. Compute the normalized $Z_{Ln}= Z_L/Z_o$
3. Locate Z_{Ln} on the chart and identify the Γ_L phasor point [1].
4. Draw a circle centered at the origin passing by Γ_L
5. Move on the circle of step 4 above "towards generator" a normalized distance of ℓ_1/λ. The new point represents Γ_1 [2]. You may read Z_{1n} and Γ_1 from this point.
6. Extend the Γ_1 phasor through the origin to reach the $-\Gamma_1$ point [3]. This is the Y_{1n} point. You may read Y_{1n} off this point.
7. Compute Y_{pn} as $1/Z_{pn}$ (or Z_o/Z_p).
8. Add Y_{pn} to Y_{1n} to obtain Y_{2n}. Locate Y_{2n} on the chart [5]. You may read $-\Gamma_2$ off that point.
9. Draw a circle centered at the origin passing by $-\Gamma_2$.
10. Extend the $-\Gamma_2$ phasor through the origin to reach the $+\Gamma_2$ point [6]. This is the Z_{2n} point. You may read Z_{2n} and Γ_2 off this point.
11. Move on the circle of step 9 above "towards generator" a normalized distance of ℓ_2/λ. The new point represents Γ_3 [7]. You may read Z_{3n} and Γ_3 from this point.
12. Compute Z_{sn} as Z_s/Z_o.
13. Add Z_{sn} to Z_{3n} to get Z_{in}. Locate Z_{in} on the chart and identify the Γ_{in} phasor point [9]. You may read Z_{inn} and Γ_{in} from this point.
14. De-normalize Z_{inn} to obtain $Z_{in}=Z_o*Z_{inn}$.

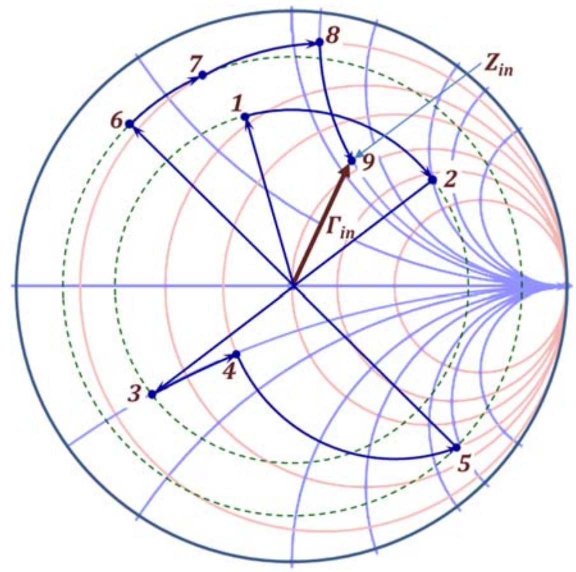


Figure 2.49

It is worth noting that the impedance and admittance additions in steps 8 and 13 above can be performed on the chart. This is simply done by the imaginary part first then the real part (or vice-versa). To add the imaginary part, you move on a constant real part circle the proper amount that corresponds to the added imaginary part. Likewise, to add the real part, you move on a constant imaginary part circle the proper amount that corresponds to the added real part. This is demonstrated in Figure 2.49 in the steps 3-4, 4-5, 7-8, and 8-9.

Smith chart Features and Short Cuts:

1. The commonly known Smith chart prints display circles and portion of circles for the $r \geq 0$. Typically, $r < 0$ (negative resistance) occurs in TL circuits with active devices. For cases requiring $r < 0$ analysis, a computer produced Smith chart is typically used.
2. A few other scales, Figure 2.50, are typically included in (commercially available) printed Smith charts; among them are:
   a. Scale for the standing wave ratio. This scale goes from 1 to ∞ corresponding to the Γ range from 0 to 1, respectively.
   b. Scales for normalized voltages and currents. These scales correspond to 1+Γ and 1-Γ. A magnitude scale for both is typically given the name "Transmission Coefficient" which ranges from 0-2. The corresponding phase angle scale, ranging from -90° to +90° is given on the inner side of the Γ phase angle circle.
3. It is important to notice that a full rotation around the chart (change of $\underline{/\Gamma}$ =2π is equivalent to a distance of λ/2. Thus, half rotation of 180° corresponds to λ/4, etc.

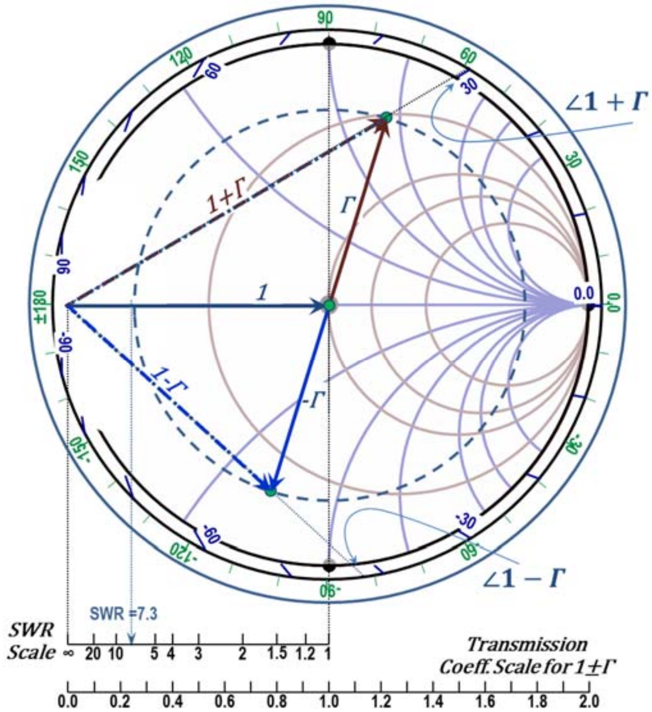


Figure 2.50

Matching using the Smith Chart:

To demonstrate the concept, let us consider the case of short-circuited shunt stub matching presented in the lower part of Figure 2.25. As you recall, the equations for this case were Equations (2.78), (2.79), and (2.81). The Smith chart offers an alternative graphical method to working out these equations analytically and numerically.

The objective is to find answers for the location of the stub on the TL, d_{ms}, as well as the stub length, d_{st}. In the following, we will demonstrate the use of the chart for this matching case. Please refer to Figure 2.51:

1. Choose the normalization impedance to be Zo to be matched to (R_s in this case).
2. Compute the normalized $Z_{Ln}= Z_L/Z_o$.
3. Locate Z_{Ln} on the chart and identify the Γ_L phasor point [1].
4. Draw a circle centered at the origin passing by Γ_L.
5. Extend the Γ_L phasor through the origin to reach the $-\Gamma_L$ point [2]. This is the Y_{Ln} point.
6. Move on the circle of step 4 above "towards generator" until it intercepts the g=1 (r=1) circle. There will be two intercept points [3] and [4]. This step is the realization of Equation (2.78) where the normalized real part of the line admittance is 1, and hence the real part of the admittance equals that of the desired matching value. Having the real part of the line admittance matched, what remains is to tune out (null) the imaginary part using the stub susceptance. Hence, the identified intercept points define the proper location for adding the shunt stub.
7. Determine the normalized distances moved from the load location to the two-intercept points: d_{ms1}/λ [5] and d_{ms2}/λ [6], respectively.

8. In principle, additional rotations of integer multiples of $\lambda/2$ will bring us to the same points, which means that we have many possible solutions in the form of $\{d_{ms1}/\lambda + 0.5\ n\}$ and $\{d_{ms2}/\lambda + 0.5\ n\}$, where n is an integer. Typically, the first two intercepts are presented as the proposed stub locations.
9. Determine the line normalized susceptances for both solutions. Notice that both values are symmetrically located and thus have the same magnitude but opposite polarities, e.g. jb_{ms1} [7] and jb_{ms2} ($-jb_{ms1}$) [8] corresponding to the d_{ms1} and d_{ms2} solutions, respectively.
10. The chosen stub is needed to null the line susceptance and hence the stub susceptance should be equal but opposite in polarity to that of the line. Hence, the desired stub susceptances should be $jB_{st1}=-(jb_{ms1}*Y_o)$ and $jB_{st2}=-(jb_{ms2}*Y_o)$, respectively.
11. Compute the normalized stub susceptances with respect to its characteristic impedance: $jb_{st1}= jB_{st1}/Y_{o,st}$ and $jb_{st2}= jB_{st2}/Y_{o,st}$, respectively.
12. To determine the proper stub length(s), we need to do another chart for the stub as a TL with a zero load impedance (infinite load admittance) and known input admittance (susceptance). This chart can be done on the same chart we used for the matching section line without interfering with the first solution. This is because the circle trace of the stub line is the circle that passes by zero load impedance (infinite admittance) which is the r=0 circle (the biggest circle on the chart). If you prefer, you may use a separate chart for the stub section.
13. Locate the zero impedance load for the stub [9]. In admittance terms, the short circuited load end is on the opposite side of the chart (the point ∞,∞) [10].
14. Locate the two possible stub susceptances on the chart, jb_{st1} and jb_{st2}, [11] and [12].
15. Starting at the (∞,∞) point and moving towards the generator, determine the normalized distances to be moved to arrive at the desired stub susceptances, d_{st1} [13] and d_{st2} [14], respectively.
16. Again, additional rotations of integer multiples of $\lambda/2$ yield valid answers which means that again we have many possible solutions in the form of $\{d_{st1}/\lambda + 0.5\ m\}$ and $\{d_{st2}/\lambda + 0.5\ m\}$, where m is an integer. Typically, the first two solutions (m=0) are presented as the proposed stub lengths.

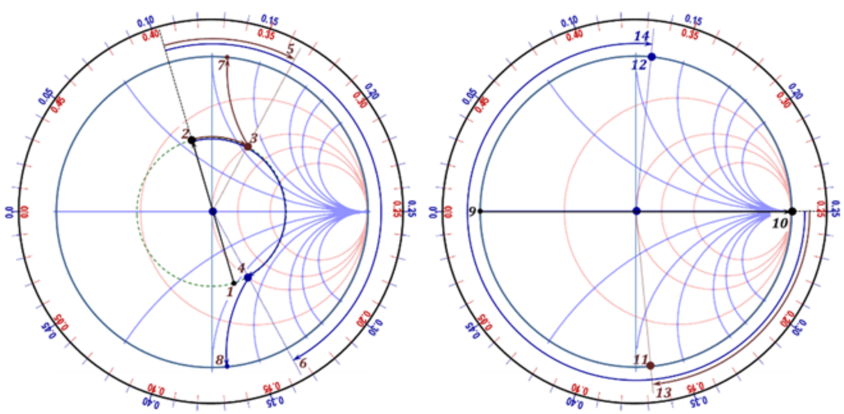


Figure 2.51